INTRODUCTION TO LOGIC

8 Identity and Definite Descriptions

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Example

Keith and Volker have the same car.
Keith and Volker have identical cars.

Keith and Volker don't share a car; they only have the same model of the same year (same colour etc).

This an example of (approximate) qualitative identity.

Qualitative identity can be formalised as a binary predicate letter expressing close similarity or sameness in all relevant aspects.

Example

This is the car that was seen at the scene.

This means probably that the *very same* car and not just a car of the same brand, the same colour etc was seen at the scene.

This is an example of numerical identity.

Occasionally it's ambiguous whether numerical or qualitative identity is meant.

In what follows I talk about numerical identity.

The language $\mathcal{L}_=$ is $\mathcal{L}_2$ plus an additional binary predicate letter = that is always interpreted as identity.

In $\mathcal{L}_2$ we can formalise 'is identical to' as a binary predicate letter, but this predicate letter can receive arbitrary relations as extension (semantic value).

In $\mathcal{L}_=$ the new binary predicate letter is always taken to express identity.

Definition (atomic formulae of $\mathcal{L}_=$)

All atomic formulae of $\mathcal{L}_2$ are atomic formulae of $\mathcal{L}_=$. Furthermore, if s and t are variables or constants, then $s = t$ is an atomic formula of $\mathcal{L}_=$.

Example

$c = a$, $x = y_3$, $x_7 = x_7$, and $x = a$ are all atomic formulae of $\mathcal{L}_=$.

The symbol '=' now plays two roles: as symbol of $\mathcal{L}_=$, but I'll continue to use it outside $\mathcal{L}_=$in the usual way.

One can use connectives and quantifiers to build formulae of $\mathcal{L}_=$ in the same ways as in $\mathcal{L}_2$.

Example

$\neg x = y$ and $\forall x(Rxy_2 \to y_2 = x)$ are formulae of $\mathcal{L}_=$.

The notion of an $\mathcal{L}_=$-sentence is defined in analogy to the notion of an $\mathcal{L}_2$-sentence.

Everything is as for $\mathcal{L}_2$, except that an additional clause needs to be added to the definition of satisfaction, where $\mathcal{A}$ is an $\mathcal{L}_2$-structure, s is a variable or constant, and t is a variable or constant:

(ix) $|s = t|^\alpha_\mathcal{A} = T$ if and only if $|s|^\alpha_\mathcal{A} = |t|^\alpha_\mathcal{A}$.

All other definitions of Chapter 5 carry over to $\mathcal{L}_=$, just with '$\mathcal{L}_2$' replaced by '$\mathcal{L}_=$'.

Caution $\mathcal{L}_=$-structures don't assign semantic values to the symbol =. There is no difference between $\mathcal{L}_=$ and $\mathcal{L}_2$-structures!

Example

$\exists x \exists y \neg x = y \nvDash \exists x \exists y Rxy$

Counterexample: Let $\mathcal{A}$ be an $\mathcal{L}_2$-structure with $\{1, 2\}$ as its domain and

$$|R^2|_\mathcal{A} = \varnothing$$

The premiss is true in exactly those $\mathcal{L}_2$-structures that have two or more elements in their domain.

Natural Deduction for $\mathcal{L}_=$ has the same rules as Natural Deduction for $\mathcal{L}_2$ except for rules for =:

=Intro

Any assumption of the form $t = t$ where t is a constant can and must be discharged.

A proof with an application of =Intro looks like this:

$$\frac{[t=t]}{\vdots}$$

=Elim

If s and t are constants, the result of appending $\phi[t/v]$ to a proof of $\phi[s/v]$ and a proof of $s = t$ or $t = s$ is a proof of $\phi[t/v]$.

$$\frac{\begin{matrix}\vdots & \vdots \\ \phi[s/v] & s=t\end{matrix}}{\phi[t/v]}\ \boxed{=\text{Elim}} \qquad \frac{\begin{matrix}\vdots & \vdots \\ \phi[s/v] & t=s\end{matrix}}{\phi[t/v]}\ \boxed{=\text{Elim}}$$

Strictly speaking, only one of the versions is needed, as from $s = t$ one can always obtain $t = s$ using only one of the rules.

Example

$\vdash \forall x\, \forall y\, (Rxy \to (x = y \to Ryx))$

Here is the proof:

Theorem (ADEQUACY)

Assume that ϕ and all elements of Γ are $\mathcal{L}_=$-sentences. Then $\Gamma \vdash \phi$ if and only if $\Gamma \vDash \phi$.

Using = one can formalise overt identity claims:

Example

William II is Wilhelm II.

FORMALISATION

$a = b$

a: William II
b: Wilhelm II

Don't confuse identity with predication.

Example

William is an emperor.

FORMALISATION

Qa

a: William
Q: … is an emperor

Here 'is' forms part of the predicate 'is an emperor'. 30

Example

William is the emperor.

Here 'is' expresses identity.

Identity can also be used in formalisations of sentences that do not involve identity explicitly.

Example

There are at least two Wagner operas.

FORMALISATION

$\exists x \exists y (Px \wedge Py \wedge \neg x = y)$

P: … is a Wagner opera

The trick works also for 'at least three' and so on.

Example

There is at most one Wagner opera.

FORMALISATION

$\forall x \forall y (Px \wedge Py \rightarrow x = y)$

The trick works also for 'at most two' and so on.

Example

There is *exactly* one Wagner opera.

FORMALISATION

$\exists x\, Px \wedge \forall x\, \forall y\, (Px \wedge Py \rightarrow x = y)$

This can be expressed shorter as

FORMALISATION

$\exists x\, (Px \wedge \forall y\, (Py \rightarrow y = x))$

Both formalisations can be shown to be equivalent using Natural Deduction.

The trick works also for 'exactly two' and so on.

Definite descriptions

The following expressions are definite descriptions:

- the present king of France
- Tim's car
- the person who has stolen a book from the library and who has forgotten his or her bag in the library

Formalising definite descriptions as constants brings various problems as the semantics of definite descriptions doesn't match the semantics of constants in $\mathcal{L}_=$.

Russell's trick

Example

Tim's car is red.

Paraphrase

Tim owns exactly one car and it is red.

FORMALISATION

$\exists x\, (Qx \wedge Rbx \wedge \forall y\, (Qy \wedge Rby \rightarrow x = y) \wedge Px)$

b: Tim
Q: … is a car
R: … owns …
P: … is red

This formalisation is much better than the formalisation of 'Tim's car' as a constant.
For instance, the following argument comes out as valid if Russell's trick is used (but not if a constant is used):

Example

Tim's car is red. Therefore there is a red car.

FORMALISATION

$\exists x(Qx \wedge Rbx \wedge \forall y(Qy \wedge Rby \rightarrow x = y) \wedge Px) \vdash \exists x(Px \wedge Qx)$

The proof is in the Manual.

So the English argument is valid in predicate logic with identity.

By using Russell's trick one can formalise definite descriptions in such way that the definite description may fail to refer to something. Constants, in contrast, are assigned objects in any $\mathcal{L}_2$-structure.

Using Russell's trick offers more ways to analyse sentences containing definite descriptions and negations.

Example

- Volker's yacht is white.
- Volker's yacht isn't white.

The first sentence is false, but is the second sentence true?

There is a reading under which both sentence are false. This reading can be made explicit in $\mathcal{L}_=$ using Russell's analysis of definite descriptions.

Example

Volker's yacht isn't white.

FORMALISATION

$$\exists x\big((Qx \land Rax) \land \forall y(Qy \land Ray \to x = y) \land \neg Px\big)$$

a:	Volker
Q:	... is a yacht
R:	... owns ...
P:	... is white

This formalisation expresses that Volker has exactly one yacht and that it isn't white.

Under this analysis 'Volker's yacht is white' and 'Volker's yacht isn't white' are both false.

Example

It's not the case (for whatever reason) that Volker's yacht is white.

I tend to understand this sentence in the following way:

FORMALISATION

$$\neg\exists x\big((Qx \land Rax) \land \forall y(Qy \land Ray \to x = y) \land Px\big)$$

Perhaps the sentence 'Volker's yacht isn't white' can be understood as saying the same; so it is ambiguous (scope ambiguity concerning $\neg$).

Logical constants

I have treated identity, the connectives and expressions like 'all' etc. as subject-independent vocabulary. Perhaps there are more such expressions:

- many, few, infinitely many
- necessarily, possibly
- it's obligatory that

At any rate the logical vocabulary of $\mathcal{L}_=$ is sufficient for analysing the validity of arguments in (large parts of) the sciences and mathematics.

Perhaps the above expressions can be analysed in $\mathcal{L}_=$ in the framework of specific theories.

The dark side

So far you have seen the logician mainly as a kind of philosophical hygienist, who makes sure that philosophers don't blunder by using logically invalid arguments or by messing up the scopes of quantifiers or connectives.

Logic seems to be an auxiliary discipline for sticklers who secure the foundations of other disciplines.

But there is also a dark side.

Here is an example. 10

Russell's paradox

If there are any safe foundations in any discipline, then the foundations of mathematics and logic should be unshakable.

I have used sets for the foundations of logic: relations, functions, $\mathcal{L}_2$-structures are defined in terms of sets. Also mathematics (and with it the sciences and various parts of philosophy) are founded on set theory. But the theory of sets is threatened by paradox.

Example (Exercise 7.6)

There is not set $\{d : d \notin d\}$ that contains exactly those things that do not have themselves as elements.

Thus defining sets using expressions $\{d : \ldots d \ldots\}$ is risky at least. So, presumably, the assumptions about sets you used at school form an inconsistent set of assumptions: anything can be proved from them.

Russell's paradox shattered Frege's foundations of mathematics.

Logicians have developed theories of sets in which the Russell paradox does not arise (e.g. Zermelo-Fraenkel set theory). Mathematicians are using the theory of sets as their foundations.

But there remained doubt in the hearts of some mathematicians and philosophers: they still didn't know that the theory of sets (and therefore the foundations of mathematics) is consistent as there could be other paradoxes.

The hope: one day a white knight would come and *prove*, using the instruments of logic, that the revised theory of sets is (syntactically or semantically) consistent. Some tried…

In the end a black knight came and, using the methods of logic, proved roughly the following:

> *If there is a proof of the consistency of set theory, using the tools of logic and set theory, then set theory is inconsistent.*

We can *never* prove, perhaps never know, that the foundations are safe (consistent). Not only did the white knights fail, they failed by necessity.

Gödel's proof is so devastatingly general that replacing set theory with a tamer theory will not help against Gödel's result. One can prove the consistency of one's standpoint only if that standpoint is inconsistent.

What remains is, perhaps, faith…