INTRODUCTION TO LOGIC

8 Identity and Definite Descriptions

Volker Halbach

> The analysis of the beginning would thus yield the notion of the unity of being and not-being – or, in a more reflected form, the unity of differentiatedness and non-differentiatedness, or the identity of identity and non-identity. *Hegel, The Science of Logic*

Assume Keith and Volker don't share a car; they only have the same model of the same year (same colour etc).

Example

Keith and Volker have the same car.
Keith and Volker have identical cars.

This an example of (approximate) qualitative identity.

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Qualitative identity can be formalised as a binary predicate letter expressing close similarity or sameness in all relevant aspects.

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This means probably that the *very same* car and not just a car of the same brand, the same colour etc was seen at the scene.

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Occasionally it's ambiguous whether numerical or qualitative identity is meant.

In what follows I talk about numerical identity.

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In $\mathcal{L}_2$ we can formalise 'is identical to' as a binary predicate letter, but this predicate letter can receive arbitrary relations as extension (semantic value).

In $\mathcal{L}_=$ the new binary predicate letter is always taken to express identity.

Definition (atomic formulae of $\mathcal{L}_=$)

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The symbol '=' now plays two roles: as symbol of $\mathcal{L}_=$ and as a symbol in the metalanguage.

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The notion of an $\mathcal{L}_=$-sentence is defined in analogy to the notion of an $\mathcal{L}_2$-sentence.

Everything is as for $\mathcal{L}_2$, except that an additional clause needs to be added to the definition of satisfaction, where $\mathcal{A}$ is an $\mathcal{L}_2$-structure, s is a variable or constant, and t is a variable or constant:

(ix) $|s=t|^{\alpha}_{\mathcal{A}} = \text{T}$ if and only if $|s|^{\alpha}_{\mathcal{A}} = |t|^{\alpha}_{\mathcal{A}}$.

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Caution: $\mathcal{L}_=$-structures don't assign semantic values to the symbol =. There is no difference between $\mathcal{L}_=$ and $\mathcal{L}_2$-structures!

Example

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Counterexample: Let $\mathcal{A}$ be any $\mathcal{L}_2$-structure with $\{1, 2\}$ as its domain.

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=Intro

Any assumption of the form $t = t$ where t is a constant can and must be discharged.

A proof with an application of =Intro looks like this:

$$\frac{[t = t]}{\vdots}$$

=Elim

If s and t are constants, the result of appending $\phi[t/v]$ to a proof of $\phi[s/v]$ and a proof of $s = t$ or $t = s$ is a proof of $\phi[t/v]$.

$$\dfrac{\begin{matrix}\vdots \\ \phi[s/v]\end{matrix} \qquad \begin{matrix}\vdots \\ s = t\end{matrix}}{\phi[t/v]}\ \boxed{=\text{Elim}} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \phi[s/v]\end{matrix} \qquad \begin{matrix}\vdots \\ t = s\end{matrix}}{\phi[t/v]}\ \boxed{=\text{Elim}}$$

Strictly speaking, only one of the versions is needed, as from $s = t$ one can always obtain $t = s$ using only one of the rules.

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Theorem (ADEQUACY)

Assume that ϕ and all elements of Γ are $\mathcal{L}_=$-sentences. Then $\Gamma \vdash \phi$ if and only if $\Gamma \vDash \phi$.

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FORMALISATION

$a = b$

a: William II
b: Wilhelm II

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FORMALISATION

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Q: … is an emperor

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Here 'is' expresses identity.

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Similar tricks work for various other numerical quantifiers 'at least three', 'at most 2', and so on.

There is no reference to numbers.

Definite descriptions

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- the person who has stolen a book from the library and who has forgotten his or her bag in the library

Formalising definite descriptions as constants brings various problems as the semantics of definite descriptions doesn't match the semantics of constants in $\mathcal{L}_=$.

Russell's trick

Example

Tim's car is red.

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Paraphrase

Tim owns exactly one car and it is red.

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FORMALISATION

$\exists x\,(Qx \wedge Rbx \wedge \forall y\,(Qy \wedge Rby \rightarrow x = y) \wedge Px)$

b: Tim

Q: … is a car

R: … owns …

P: … is red

This formalisation is much better than the formalisation of 'Tim's car' as a constant.
For instance, the following argument comes out as valid if Russell's trick is used (but not if a constant is used):

Example

Tim's car is red. Therefore there is a red car.

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$$\exists x(Qx \land Rbx \land \forall y(Qy \land Rby \to x = y) \land Px) \vdash \exists x(Px \land Qx)$$

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So the English argument is valid in predicate logic with identity.

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- Volker's Ferrari is red.
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The first sentence is false, but is the second sentence true?

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The first sentence is false, but is the second sentence true?

There is a reading under which both sentence are false. This reading can be made explicit in $\mathcal{L}_=$ using Russell's analysis of definite descriptions.

Example

Volker's Ferrari isn't red.

FORMALISATION

$$\exists x\big((Qx \wedge Rax) \wedge \forall y(Qy \wedge Ray \to x = y) \wedge \neg Px\big)$$

a: Volker
Q: … is a Ferrari
R: … owns …
P: … is red

This formalisation expresses that Volker has exactly one Ferrari and that it isn't red.

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This formalisation expresses that Volker has exactly one Ferrari and that it isn't red.

Under this analysis 'Volker's Ferrari is red' and 'Volker's Ferrari isn't red' are both false.

Example

It's not the case (for whatever reason) that Volker's Ferrari is red.

I tend to understand this sentence in the following way:

FORMALISATION

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Logical constants

I have treated identity, the connectives and expressions like ‘all’ etc. as subject-independent vocabulary. Perhaps there are more such expressions:

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At any rate the logical vocabulary of $\mathcal{L}_=$ is sufficient for analysing the validity of arguments in (large parts of) the sciences and mathematics.

Perhaps the above expressions can be analysed in $\mathcal{L}_=$ in the framework of specific theories.

The dark side

So far you have seen the logician mainly as a kind of philosophical hygienist, who makes sure that philosophers don't blunder by using logically invalid arguments or by messing up the scopes of quantifiers or connectives.

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Russell's paradox

If there are any safe foundations in any discipline, then the foundations of mathematics and logic should be unshakable.

I have used sets for the foundations of logic: sets, relations, and functions. $\mathcal{L}_2$-structures are defined in terms of sets. Large parts of various disciplines (mathematics, sciences, various parts of philosophy) are founded on set theory.

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Sets replace in many cases the role assigned to universals in classical philosophy.

But the theory of sets is threatened by paradox.

Example (Exercise 7.6)

There is not set $\{d : d \notin d\}$ that contains exactly those things that do not have themselves as elements.

Thus defining sets using expressions $\{d : \ldots d \ldots\}$ is risky. So presumably the assumptions about sets you used at school form an inconsistent set of assumptions: anything can be proved from them.

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But there remained doubt in the hearts of some mathematicians and philosophers: they still didn't know that the theory of sets (and therefore the foundations of mathematics) is consistent as there could be other paradoxes.

The hope: one day a white knight would come and *prove*, using the instruments of logic, that the revised theory of sets is (syntactically or semantically) consistent. Some tried…

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What remains is, perhaps, faith…