INTRODUCTION TO LOGIC

7 Formalisation in Predicate Logic

Volker Halbach

‘Contrariwise,’ continued Tweedledee, ‘if it was so, it might be; and if it were so, it would be; but as it isn’t, it ain’t. That’s logic.’

Lewis Carroll, Through the Looking-Glass, Chapter 4

In this lecture I'll wrap up my treatment of predicate logic by bringing together three strands:

- predicate logic and English (chapter 4)
- the semantics of predicate logic (chapter 5)
- Natural Deduction (chapter 6)

Theorem (adequacy)

Assume that ϕ and all elements of Γ are $\mathcal{L}_2$-sentences. Then $\Gamma \vdash \phi$ if and only if $\Gamma \vDash \phi$.

Consistency

Definition (syntactic consistency)

A set Γ of $\mathcal{L}_2$-sentences is syntactically consistent iff there is a sentence ϕ such that $\Gamma \nvdash \phi$.

A set Γ is syntactically inconsistent iff it's not syntactically consistent.

FIRST REMARK

A set Γ is syntactically inconsistent iff for all sentences ϕ of $\mathcal{L}_2$, $\Gamma \vdash \phi$.

FIRST REMARK

A set Γ is syntactically inconsistent iff for all sentences ϕ of $\mathcal{L}_2$, $\Gamma \vdash \phi$.

SECOND REMARK

A set Γ is syntactically inconsistent iff $\Gamma \vdash (P \wedge \neg P)$.

Here P is the sentence letter (I could have used any other sentence). To show that the first remark follows from the second, one proves that $\Gamma \vdash \phi$ for any sentence ϕ if $\Gamma \vdash (P \wedge \neg P)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ P \wedge \neg P\end{array}}{P} \qquad \dfrac{\begin{array}{c}\vdots\\ P \wedge \neg P\end{array}}{\neg P}}{\exists x \exists y\, Rxy}\ \neg\text{Elim}$$

Definition (semantic consistency, chapter 5)

A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if there is an $\mathcal{L}_2$-structure $\mathcal{A}$ in which all sentences in Γ are true.

Definition (semantic consistency, chapter 5)

A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if there is an $\mathcal{L}_2$-structure $\mathcal{A}$ in which all sentences in Γ are true.

Theorem (using the adequacy theorem)

A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if Γ is syntactically consistent.

Definition (semantic consistency, chapter 5)

A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if there is an $\mathcal{L}_2$-structure $\mathcal{A}$ in which all sentences in Γ are true.

Theorem (using the adequacy theorem)

A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if Γ is syntactically consistent.

In order to show that a set of sentences is semantically or syntactically consistent, one can prove that there is an $\mathcal{L}_2$-structure in which all sentences in the set are true.

Definition (semantic consistency, chapter 5)

A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if there is an $\mathcal{L}_2$-structure $\mathcal{A}$ in which all sentences in Γ are true.

Theorem (using the adequacy theorem)

A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if Γ is syntactically consistent.

In order to show that a set of sentences is semantically or syntactically consistent, one can prove that there is an $\mathcal{L}_2$-structure in which all sentences in the set are true.

In order to show that a set Γ of sentences is inconsistent, one can prove that $\Gamma \vdash (P \wedge \neg P)$. For any inconsistent set there is such a proof.

Definition (semantic consistency, chapter 5)

A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if there is an $\mathcal{L}_2$-structure $\mathcal{A}$ in which all sentences in Γ are true.

Theorem (using the adequacy theorem)

A set Γ of $\mathcal{L}_2$-sentences is semantically consistent if and only if Γ is syntactically consistent.

In order to show that a set of sentences is semantically or syntactically consistent, one can prove that there is an $\mathcal{L}_2$-structure in which all sentences in the set are true.

In order to show that a set Γ of sentences is inconsistent, one can prove that $\Gamma \vdash (P \wedge \neg P)$. For any inconsistent set there is such a proof.

For finite sets of $\mathcal{L}_1$-sentences we have the truth table method.

Decidability

In contrast to $\mathcal{L}_1$, we still don't have a systematic method for checking whether an argument in $\mathcal{L}_2$ (with finitely many premisses) is valid or whether an $\mathcal{L}_2$-sentence is a logical truth or whether it is inconsistent.

Decidability

In contrast to $\mathcal{L}_1$, we still don't have a systematic method for checking whether an argument in $\mathcal{L}_2$ (with finitely many premisses) is valid or whether an $\mathcal{L}_2$-sentence is a logical truth or whether it is inconsistent.

Theorem (Church 1936)

There is no 'recursive' method for deciding whether an $\mathcal{L}_2$-sentence is logically true (or whether an $\mathcal{L}_2$-argument with finitely many premisses is valid).

That is, one cannot write a computer programme that tells one, applied to an $\mathcal{L}_2$-sentence, after finite time whether the sentence is logically true or not. That holds even if no restrictions are imposed on the memory, disk space, computation time etc.

Consequently, there is no method for deciding whether a given $\mathcal{L}_2$-sentence is provable.

How does the formal language $\mathcal{L}_2$ relate to English? In chapter 4 I have already sketched how one goes about formalisations of English sentences in $\mathcal{L}_2$.

How does the formal language $\mathcal{L}_2$ relate to English? In chapter 4 I have already sketched how one goes about formalisations of English sentences in $\mathcal{L}_2$.

Definition

An argument in English is valid in predicate logic if and only if its formalisation in the language $\mathcal{L}_2$ of predicate logic is valid.

Example

All concrete objects are located in space. The number 5 isn't located in space. So the number 5 isn't a concrete object.

FORMALISATION

$\forall x\,(Px \to Qx), \neg Qa \vdash \neg Pa$

P: … is a concrete object
Q: … is located in space
a: the number 5

Example

All concrete objects are located in space. The number 5 isn't located in space. So the number 5 isn't a concrete object.

FORMALISATION

$\forall x\,(Px \to Qx), \neg Qa \vdash \neg Pa$

- P: … is a concrete object
- Q: … is located in space
- a: the number 5

$$\forall x(Px \to Qx)$$

Example

All concrete objects are located in space. The number 5 isn't located in space. So the number 5 isn't a concrete object.

FORMALISATION

$\forall x\,(Px \to Qx), \neg Qa \vdash \neg Pa$

P: … is a concrete object
Q: … is located in space
a: the number 5

$$\frac{\forall x(Px \to Qx)}{Pa \to Qa}$$

Example

All concrete objects are located in space. The number 5 isn't located in space. So the number 5 isn't a concrete object.

FORMALISATION

$\forall x\,(Px \to Qx), \neg Qa \vdash \neg Pa$

P: … is a concrete object
Q: … is located in space
a: the number 5

$$Pa \qquad \frac{\forall x(Px \to Qx)}{Pa \to Qa}$$

Example

All concrete objects are located in space. The number 5 isn't located in space. So the number 5 isn't a concrete object.

FORMALISATION

$\forall x\,(Px \to Qx), \neg Qa \vdash \neg Pa$

P: … is a concrete object
Q: … is located in space
a: the number 5

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to Qx)}{Pa \to Qa}}{Qa}$$

Example

All concrete objects are located in space. The number 5 isn't located in space. So the number 5 isn't a concrete object.

FORMALISATION

$\forall x\,(Px \to Qx), \neg Qa \vdash \neg Pa$

P:	… is a concrete object
Q:	… is located in space
a:	the number 5

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to Qx)}{Pa \to Qa}}{Qa} \qquad\qquad \neg Qa$$

Example

All concrete objects are located in space. The number 5 isn't located in space. So the number 5 isn't a concrete object.

FORMALISATION

$\forall x\,(Px \to Qx), \neg Qa \vdash \neg Pa$

- P: … is a concrete object
- Q: … is located in space
- a: the number 5

$$\dfrac{\dfrac{[Pa] \qquad \dfrac{\forall x(Px \to Qx)}{Pa \to Qa}}{Qa} \qquad \neg Qa}{\neg Pa}$$

So the argument is valid in predicate logic.

Above I formalised ‘is located in space’ by a single predicate letter rather than a binary one and a constant.

When showing that an argument is valid in predicate logic, one doesn’t have to give a *full* formalisation when one is able to prove the validity of a partial formalisation.

Above I formalised ‘is located in space’ by a single predicate letter rather than a binary one and a constant.

When showing that an argument is valid in predicate logic, one doesn’t have to give a *full* formalisation when one is able to prove the validity of a partial formalisation.

Caution When showing that an argument is *not* valid in predicate logic you need to give the full formalisation (because a more detailed formalisation might yield a valid argument).

Example: show the following argument is not valid

A belief is known only if it is true and justified. The belief that Jones is in Barcelona or Jones owns a Ford is true and justified. Therefore, it's known.

Example: show the following argument is not valid

A belief is known only if it is true and justified. The belief that Jones is in Barcelona or Jones owns a Ford is true and justified. Therefore, it's known.

Step (i): formalise

Premiss 1: $\forall x\,(Px \to (P_1x \to (Qx \land Rx)))$.

Dictionary:

P:	… is a belief
P_1:	… is known
Q:	… is true
R:	… is justified
a:	the belief that Jones is in Barcelona or Jones owns a Ford

Example: show the following argument is not valid

A belief is known only if it is true and justified. The belief that Jones is in Barcelona or Jones owns a Ford is true and justified. Therefore, it's known.

Step (i): formalise

Premiss 1: $\forall x\,(Px \to (P_1x \to (Qx \wedge Rx)))$.
Premiss 2: $Pa \wedge Qa \wedge Ra$.

Dictionary:

P: … is a belief
P_1: … is known
Q: … is true
R: … is justified
a: the belief that Jones is in Barcelona or Jones owns a Ford

Example: show the following argument is not valid

A belief is known only if it is true and justified. The belief that Jones is in Barcelona or Jones owns a Ford is true and justified. Therefore, it's known.

Step (i): formalise

Premiss 1: $\forall x\,(Px \to (P_1x \to (Qx \wedge Rx)))$.
Premiss 2: $Pa \wedge Qa \wedge Ra$.
Conclusion: P_1a.

Dictionary:

- P: … is a belief
- P_1: … is known
- Q: … is true
- R: … is justified
- a: the belief that Jones is in Barcelona or Jones owns a Ford

Claim

$\forall x\,(Px \to (P_1 x \to Qx \wedge Rx)), Pa \wedge Qa \wedge Ra \nvDash P_1 a$

(Because of the adequacy theorem $\nvdash$ and $\nvDash$ coincide.)

Claim

$\forall x\,(Px \to (P_1x \to Qx \land Rx)), Pa \land Qa \land Ra \nvDash P_1a$

(Because of the adequacy theorem $\nvdash$ and $\nvDash$ coincide.)

Here is a counterexample:
Let $\mathcal{A}$ be an $\mathcal{L}_2$-structure with $\{1\}$ as its domain and

$$
\begin{aligned}
|P|_{\mathcal{A}} &= \{1\} \\
|P_1|_{\mathcal{A}} &= \varnothing \\
|Q|_{\mathcal{A}} &= \{1\} \\
|R|_{\mathcal{A}} &= \{1\} \\
|a|_{\mathcal{A}} &= 1
\end{aligned}
$$

The premisses are true, the conclusion is false in this structure.

Logical truth of English sentences in predicate logic etc. are defined in analogy to the notions of logical truth etc. in propositional logic:

Definition

1. An English sentence is logically true in predicate logic iff its formalisation in predicate logic is logically true.
2. An English sentence is a contradiction in predicate logic iff its formalisation in predicate logic is a contradiction.
3. A set of English sentences is consistent in predicate logic iff the set of their formalisations in predicate logic is semantically consistent.

To show that an English sentence is logically true in predicate logic, one can (try to) formalise the sentence as a sentence ϕ of $\mathcal{L}_2$ and prove that $\vdash \phi$.

To show that an English sentence is logically true in predicate logic, one can (try to) formalise the sentence as a sentence ϕ of $\mathcal{L}_2$ and prove that $\vdash \phi$.

To show that an English sentence is a contradiction in predicate logic one can formalise the sentence as a sentence ϕ of $\mathcal{L}_2$ and prove that $\vdash \neg\phi$.

To show that an English sentence is logically true in predicate logic, one can (try to) formalise the sentence as a sentence ϕ of $\mathcal{L}_2$ and prove that $\vdash \phi$.

To show that an English sentence is a contradiction in predicate logic one can formalise the sentence as a sentence ϕ of $\mathcal{L}_2$ and prove that $\vdash \neg\phi$.

To show that a set of English sentences is consistent in predicate logic, one can formalise all sentences in the set and show that the formalisations are are all true in *some* $\mathcal{L}_2$-structure.

To show that an English sentence is logically true in predicate logic, one can (try to) formalise the sentence as a sentence ϕ of $\mathcal{L}_2$ and prove that $\vdash \phi$.

To show that an English sentence is a contradiction in predicate logic one can formalise the sentence as a sentence ϕ of $\mathcal{L}_2$ and prove that $\vdash \neg\phi$.

To show that a set of English sentences is consistent in predicate logic, one can formalise all sentences in the set and show that the formalisations are are all true in *some* $\mathcal{L}_2$-structure.

To show that a set of English sentences is inconsistent in predicate logic, one can formalise some of the sentences as $\phi_1, \ldots, \phi_n$ and show that $\{\phi_1, \ldots, \phi_n\} \vdash (P \wedge \neg P)$.

The language $\mathcal{L}_2$ is very powerful. Large parts of the sciences and mathematics can easily be formalised in it.

The language $\mathcal{L}_2$ is very powerful. Large parts of the sciences and mathematics can easily be formalised in it.

I return to the problem of formalising English sentences in $\mathcal{L}_2$. As we can now analyse more structural features of English sentences, we get new problems.

FIRST PROBLEM

Arity of predicates

FIRST PROBLEM

Arity of predicates

Jones buttered the toast with a knife in the bathroom.
Jones buttered the toast with a knife.
Jones buttered the toast.

Should one formalise the predicate as a predicate letter of arity 4, 3, or 2?

FIRST PROBLEM

Arity of predicates

Jones buttered the toast with a knife in the bathroom.
Jones buttered the toast with a knife.
Jones buttered the toast.

Should one formalise the predicate as a predicate letter of arity 4, 3, or 2?

Arguably, one could paraphrase the last sentence as

Jones buttered the toast with something in some place.

and then use a predicate letter of arity 4 for the formalisation.

SECOND PROBLEM

Lexical ambiguity

SECOND PROBLEM

Lexical ambiguity

The predicate expressions 'is a bank', 'is a suit' are ambiguous.

In $\mathcal{L}_2$ there is no lexical ambiguity: in a $\mathcal{L}_2$-structure the semantic value of a unary predicate letter is always a single set. Accordingly, one has to use different predicate letters for 'is a bank' (as a financial institution) and for 'is a bank' (as the edge of a river).

THIRD PROBLEM

Structural ambiguity

THIRD PROBLEM

Structural ambiguity

The indefinite article can be used to make existential or universal claims.

Example

A New College student is clever.

THIRD PROBLEM

Structural ambiguity

The indefinite article can be used to make existential or universal claims.

Example

A New College student is clever.

This is ambiguous between:

1. $\exists x\,(Px \wedge Qx)$ and
2. $\forall x\,(Px \rightarrow Qx)$

P: … is a New College student
Q: … is clever

Example

All the books were taken by someone.

Example

All the books were taken by someone.

Arguably, there are two readings:

1. $\forall y\,(Py \to \exists x\,(Qx \wedge Rxy))$ and
2. $\exists x(Qx \wedge \forall y\,(Py \to Rxy))$

P: … is a book
Q: … is a person
R: … took …

The kind of ambiguity in

All the books were taken by someone.

is known as scope ambiguity because the formalisations assign different *scopes* to the quantifier $\forall x$.

Definition (scope of a quantifier)

The scope of an occurrence of a quantifier in a sentence ϕ is (the occurrence of) the smallest $\mathcal{L}_2$-formula that contains that quantifier and is part of ϕ.

FOURTH PROBLEM

Intensionality

FOURTH PROBLEM

Intensionality

Example

Sören believes in an almighty being. Therefore there is an almighty being.

INCORRECT FORMALISATION, Exercise 6.3 (i)

$\exists x\,(Rax \wedge Px) \vdash \exists x\, Px$

a: Sören
P: … is an almighty being
R: … believes in …

INCORRECT FORMALISATION

$\exists x\,(Rax \wedge Px) \vdash \exists x\,Px$

INCORRECT FORMALISATION

$\exists x\,(Rax \wedge Px) \vdash \exists x\,Px$

$$\exists x\,(Rax \wedge Px)$$

INCORRECT FORMALISATION

$\exists x\,(Rax \land Px) \vdash \exists x\,Px$

$$Rab \land Pb$$

$$\exists x\,(Rax \land Px)$$

INCORRECT FORMALISATION

$\exists x\,(Rax \wedge Px) \vdash \exists x\, Px$

$$\exists x\,(Rax \wedge Px) \qquad \dfrac{Rab \wedge Pb}{Pb}$$

INCORRECT FORMALISATION

$\exists x\,(Rax \wedge Px) \vdash \exists x\, Px$

$$\exists x\,(Rax \wedge Px) \qquad \dfrac{\dfrac{Rab \wedge Pb}{Pb}}{\exists x\, Px}$$

INCORRECT FORMALISATION

$\exists x\,(Rax \wedge Px) \vdash \exists x\, Px$

$$\dfrac{\exists x\,(Rax \wedge Px) \qquad \dfrac{\dfrac{[Rab \wedge Pb]}{Pb}}{\exists x\, Px}}{\exists x\, Px}$$

INCORRECT FORMALISATION

$\exists x\,(Rax \wedge Px) \vdash \exists x\,Px$

$$\dfrac{\exists x\,(Rax \wedge Px) \qquad \dfrac{\dfrac{[Rab \wedge Pb]}{Pb}}{\exists x\,Px}}{\exists x\,Px}$$

So the $\mathcal{L}_2$-argument is valid.

Probably this is not a good proof for the existence of an almighty being.

Probably this is not a good proof for the existence of an almighty being.

What's going wrong here?

Probably this is not a good proof for the existence of an almighty being.

What's going wrong here?

The English predicate expression 'believes in' doesn't express a relation between the believer and another object. It's semantics is different from the semantics of a predicate letter of $\mathcal{L}_2$.

Example

Miles wants to live in Oxford. Oxford is a city with high levels of air pollution. Therefore Miles wants to live in a city with high levels of air pollution.

INCORRECT FORMALISATION

Rab

Qb

$\exists x\,(Rax \wedge Qx)$

R: … wants to live in …

Q: … is a city with high levels of air pollution

a: Miles

b: Oxford

If the formalisation were sound the argument would be valid in predicate logic as

Claim

$Rab, Qb \vdash \exists x\,(Rax \land Qx)$

If the formalisation were sound the argument would be valid in predicate logic as

Claim

$Rab, Qb \vdash \exists x\,(Rax \wedge Qx)$

$$\dfrac{\dfrac{Rab \qquad Qb}{Rab \wedge Qb}}{\exists x\,(Rax \wedge Qx)}\ \exists\text{Intro}$$

Again ‘wants to live in’ doesn’t express a relation between a person and a place. Hence it cannot be formalised using a binary predicate letter (at best one can formalise ‘wants to live in Oxford’ and ‘wants to live in a city with high levels of air pollution’ as two separate unary predicate letters.

Again ‘wants to live in’ doesn’t express a relation between a person and a place. Hence it cannot be formalised using a binary predicate letter (at best one can formalise ‘wants to live in Oxford’ and ‘wants to live in a city with high levels of air pollution’ as two separate unary predicate letters.

This shows that in some cases the truth of an English sentence does not only depend on what a designator designates.

Again ‘wants to live in’ doesn’t express a relation between a person and a place. Hence it cannot be formalised using a binary predicate letter (at best one can formalise ‘wants to live in Oxford’ and ‘wants to live in a city with high levels of air pollution’ as two separate unary predicate letters.

This shows that in some cases the truth of an English sentence does not only depend on what a designator designates.

This is not the case in $\mathcal{L}_2$.

Extensionality of $\mathcal{L}_2$

If constants, sentence letters, and predicate letters are replaced in an $\mathcal{L}_2$-sentence by other constants, sentence letters, and predicate letters (respectively) that have the same extension in a given $\mathcal{L}_2$-structure, then the truth-value of the sentence in that $\mathcal{L}_2$-structure does not change.

That is, the truth value of an $\mathcal{L}_2$-sentence in an $\mathcal{L}_2$-structure depends only on the extension (semantic value) of the symbols in the sentence in that $\mathcal{L}_2$-structure.

Extensionality of $\mathcal{L}_2$

If constants, sentence letters, and predicate letters are replaced in an $\mathcal{L}_2$-sentence by other constants, sentence letters, and predicate letters (respectively) that have the same extension in a given $\mathcal{L}_2$-structure, then the truth-value of the sentence in that $\mathcal{L}_2$-structure does not change.

That is, the truth value of an $\mathcal{L}_2$-sentence in an $\mathcal{L}_2$-structure depends only on the extension (semantic value) of the symbols in the sentence in that $\mathcal{L}_2$-structure.

In contrast to English, $\mathcal{L}_2$ is an extensional language.

'that'-sentences and ontology

Example

Fred believes that 8 is (identical to) 8.
Fred believes that the number of planets is 8.

‘that’-sentences and ontology

Example

Fred believes that 8 is (identical to) 8.
Fred believes that the number of planets is 8.

Example

It’s necessary that 8 is 8.
It’s necessary that the number of planets is 8.

‘that’-sentences and ontology

Example

Fred believes that 8 is (identical to) 8.
Fred believes that the number of planets is 8.

Example

It’s necessary that 8 is 8.
It’s necessary that the number of planets is 8.

These two examples show that

> *… believes that … is …*
> *it’s necessary that … is …*

do not express relations and thus *must not* be formalised as predicate letters.

Some philosophers have proposed to analyse these sentences using propositions:

Example

Fred believes the proposition that 8 is 8.

formalisation

R_1ab

R_1: … believes …
a: Fred
b: the proposition that 8 is 8

Some philosophers have proposed to analyse these sentences using propositions:

Example

Fred believes the proposition that 8 is 8.

formalisation

R_1ab

R_1:	… believes …
a:	Fred
b:	the proposition that 8 is 8

In more sophisticated formalisations the constant b might be further analysed (even within predicate logic). At any rate we are now getting into metaphysical problems: What are propositions (if they exist at all)? How are propositions structured?

Different philosophical views force different formalisations:

- If belief is a relation between a believer and an proposition, the formalisation R_1ab is adequate.
- If belief is merely a certain state of mind and the believer is not entering a relation with some object (proposition etc), then R_1ab is surely not adequate.

Quotation

The phrase

'. . . ' has six letters

is not extensional.

Quotation

The phrase

> *'. . . ' has six letters*

is not extensional.

Example

'London' has six letters.
'the capital of England' has six letters.

Looking back to Chapter 1 should shed some light on how to deal with quotation marks in formalisations.

Looking back to Chapter 1 should shed some light on how to deal with quotation marks in formalisations.

Example

'snow' is a noun.

This isn't a sentence about snow; this is a sentence about the *word* 'snow'. Thus, the above sentence *must not* be formalised as Pa with the following dictionary:

P: '…' is a noun

a: snow

But it can be formalised as Qb.

Q: … is a noun

b: 'snow'

The *spoken* sentence

> *Tom is monosyllabic.*

is ambiguous. The ambiguity is made explicit by the following two formalisations.

FORMALISATION I

Qa

Q: … is monosyllabic
a: 'Tom'

FORMALISATION II

Qb

Q: … is monosyllabic
b: Tom

One might argue that 'is monosyllabic' is ambiguous at least in spoken English.