INTRODUCTION TO LOGIC

5 The Semantics of Predicate Logic

Volker Halbach

We could forget about philosophy. Settle down and maybe get into semantics.

Woody Allen, *Mr. Big*

Outline

1. Validity.
2. Semantics for simple English sentences.
3. Semantics for $\mathcal{L}_2$-formulae.
4. $\mathcal{L}_2$-structures.

Argument **Valid**

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What is it for this $\mathcal{L}_2$-argument to be valid?

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It remains to define: *$\mathcal{L}_2$-structure*, *truth in an $\mathcal{L}_2$-structure*

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- An $\mathcal{L}_2$-structure $\mathcal{A}$ assigns each predicate and constant a semantic value (specifically, what?).

I could present all definitions on 4 slides. Most slides just help to motivate these definitions.

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Notation

When e is an expression, we write $|e|$ for its semantic value.

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In other words:

|‘Mary likes Maggie Smith’| = T iff
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We'll take this one step further, by saying more about properties and relations.

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Example

The property of *being an actor*

$$= \text{the set of actors}$$
$$= \{d : d \text{ is an actor}\}$$
$$= \{\text{Emma Stone, B. Cumberbatch, } \ldots\}$$

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Notation: $|e|_{\mathcal{A}}$ is the semantic value of e in $\mathcal{L}_2$-structure $\mathcal{A}$.

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What object each variable denotes is specified with a *variable assignment*.

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Similarly for other atomic formulae.

Worked example

Let $\mathcal{L}_2$-structure $\mathcal{A}$ be such that:

- $|a|_{\mathcal{A}} = \text{Venus}$
- $|b|_{\mathcal{A}} = \text{Mars}$
- $|P^1|_{\mathcal{A}} = \{\text{Saturn, Mars}\}$
- $|R^2|_{\mathcal{A}} = \{\langle\text{Venus, Mars}\rangle\}$

Let assignments α and β be such that:

	x	y	z
α:	Saturn	Mars	Jupiter
β:	Venus	Venus	Venus

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Compute the following:

$\lvert x\rvert_{\mathcal{A}}^{\alpha} =$	$\lvert x\rvert_{\mathcal{A}}^{\beta} =$	$\lvert a\rvert_{\mathcal{A}}^{\alpha} =$
$\lvert Py\rvert_{\mathcal{A}}^{\alpha} =$	$\lvert Py\rvert_{\mathcal{A}}^{\beta} =$	$\lvert Pb\rvert_{\mathcal{A}}^{\alpha} =$
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Let assignments α and β be such that:

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α:	**Saturn**	Mars	Jupiter
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Compute the following:

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Let assignments α and β be such that:

	x	y	z
α:	Saturn	Mars	Jupiter
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Compute the following:

$\lvert x\rvert^{\alpha}_{\mathcal{A}} =$ Saturn	$\lvert x\rvert^{\beta}_{\mathcal{A}} =$ Venus	$\lvert a\rvert^{\alpha}_{\mathcal{A}} =$
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Semantics for quantifiers

Whether the following sentence is true depends on which things there are:

> Everything is material.

Thus the truth of sentences depends on which objects there are and this needs to be taken into account in determining truth values.

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This is correct but the general case is more complex.

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To progress any further we need to be able evaluate $\exists y Rxy$ *under an assignment* α *of an object to* x.

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Say that β *differs from* α *in* y *at most* if $|v|^{\alpha} = |v|^{\beta}$ for all variables v with the possible exception of y.

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$\mathcal{L}_2$-*expression*	*semantic value in* $\mathcal{A}$
constant: a	object: $\lvert a\rvert_{\mathcal{A}}$ in $D_{\mathcal{A}}$
sentence letter: P	truth-value: $\lvert P\rvert_{\mathcal{A}}$ (= T or F)
unary predicate letter: P^1	unary relation: $\lvert P^1\rvert_{\mathcal{A}}$ (i.e. a set)
binary predicate letter: P^2	binary relation: $\lvert P^2\rvert_{\mathcal{A}}$ (a set of pairs)
ternary predicate letter: P^3	ternary relation: $\lvert P^3\rvert_{\mathcal{A}}$ (a set of triples)
etc.	

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The semantics for connectives are just like those for $\mathcal{L}_1$.

Semantics for connectives

(ii) $|\neg\phi|^{\alpha}_{\mathcal{A}} = T$ if and only if $|\phi|^{\alpha}_{\mathcal{A}} = F$.

(iii) $|\phi \wedge \psi|^{\alpha}_{\mathcal{A}} = T$ if and only if $|\phi|^{\alpha}_{\mathcal{A}} = T$ and $|\psi|^{\alpha}_{\mathcal{A}} = T$.

(iv) $|\phi \vee \psi|^{\alpha}_{\mathcal{A}} = T$ if and only if $|\phi|^{\alpha}_{\mathcal{A}} = T$ or $|\psi|^{\alpha}_{\mathcal{A}} = T$.

(v) $|\phi \rightarrow \psi|^{\alpha}_{\mathcal{A}} = T$ if and only if $|\phi|^{\alpha}_{\mathcal{A}} = F$ or $|\psi|^{\alpha}_{\mathcal{A}} = T$.

(vi) $|\phi \leftrightarrow \psi|^{\alpha}_{\mathcal{A}} = T$ if and only if $|\phi|^{\alpha}_{\mathcal{A}} = |\psi|^{\alpha}_{\mathcal{A}}$.

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However, we lack a simple decision procedure (in contrast to $\mathcal{L}_1$ and the truth table method).

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Now you know what truth is.

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Sentences of $\mathcal{L}_1$ are built up from other sentences:

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Definition

Let Γ be a set of sentences of $\mathcal{L}_2$ and ϕ a sentence of $\mathcal{L}_2$. The argument with all sentences in Γ as premisses and ϕ as conclusion is valid if and only if there is no $\mathcal{L}_2$-structure in which all sentences in Γ are true and ϕ is false.

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That the argument with all sentences in Γ as premisses and ϕ as conclusion is valid, is abbreviated as $\Gamma \vDash \phi$.

Thus, $\Gamma \vDash \phi$ iff there is no $\mathcal{L}_2$-structure such that $|\phi|_{\mathcal{A}} = \text{F}$ and for all sentences γ in Γ, $|\gamma|_{\mathcal{A}} = \text{T}$.

In general, it's difficult to prove that an argument in $\mathcal{L}_2$ is valid by proving a claim about all $\mathcal{L}_2$-structures as there is no method to go through *all* $\mathcal{L}_2$-structures.

This is in contrast to $\mathcal{L}_1$ where one can systematically check out all $\mathcal{L}_1$-structures using truth tables.

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In order to show that an argument in $\mathcal{L}_2$ is *not* valid, one can specify an $\mathcal{L}_2$-structure in which all premisses are true and the conclusion is false. Such an $\mathcal{L}_2$-structure is called a counterexample to the argument.

Example

$\forall x\,(P^1x \to Q^1x) \nvDash \forall x\,(\neg P^1x \to \neg Q^1x)$

The symbol $\nvDash$ is used to claim that the argument is *not* valid.

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The symbol $\nvDash$ is used to claim that the argument is *not* valid.

Let $\mathcal{B}$ be an $\mathcal{L}_2$-structure with $\{\text{Oxford}\}$ as its domain and

$$|P^1|_{\mathcal{A}} = \varnothing$$
$$|Q^1|_{\mathcal{A}} = \{\text{Oxford}\}$$

What $\mathcal{B}$ assigns to other constants and predicate letters doesn't matter.

Claim

$\mathcal{B}$ is a counterexample to the argument.

At first I show that the premiss is true in $\mathcal{B}$. Let α be any variable assignment over $\mathcal{B}$.

$$|x|_{\mathcal{B}}^{\alpha} \notin \varnothing$$
$$|x|_{\mathcal{B}}^{\alpha} \notin |P^1|_{\mathcal{B}}$$
$$|P^1x|_{\mathcal{B}}^{\alpha} = \mathrm{F}$$
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So $|P^1 x \to Q^1 x|_{\mathcal{B}}^{\alpha} = \mathrm{T}$ for all variable assignments α over $\mathcal{B}$

At first I show that the premiss is true in $\mathcal{B}$. Let α be any variable assignment over $\mathcal{B}$.

$$|x|^{\alpha}_{\mathcal{B}} \notin \varnothing$$
$$|x|^{\alpha}_{\mathcal{B}} \notin |P^1|_{\mathcal{B}}$$
$$|P^1 x|^{\alpha}_{\mathcal{B}} = \mathrm{F}$$
$$|P^1 x \to Q^1 x|^{\alpha}_{\mathcal{B}} = \mathrm{T}$$

So $|P^1 x \to Q^1 x|^{\alpha}_{\mathcal{B}} = \mathrm{T}$ for all variable assignments α over $\mathcal{B}$ and therefore

$$|\forall x\,(P^1 x \to Q^1 x)|_{\mathcal{B}} = \mathrm{T}$$

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So the premiss is true in $\mathcal{B}$.

I still need to show that $\forall x\,(\neg P^1 x \to \neg Q^1 x)$ is false in $\mathcal{B}$. Let β be a variable assignment over $\mathcal{B}$. Then $|x|^\beta_{\mathcal{B}} = \text{Oxford}$.

$$|x|^\beta_{\mathcal{B}} \notin \varnothing$$

$$|x|^\beta_{\mathcal{B}} \notin |P^1|_{\mathcal{B}}$$

$$|P^1 x|^\beta_{\mathcal{B}} = \text{F}$$

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I still need to show that $\forall x\,(\neg P^1 x \to \neg Q^1 x)$ is false in $\mathcal{B}$. Let β be a variable assignment over $\mathcal{B}$. Then $|x|^{\beta}_{\mathcal{B}} = \text{Oxford}$.

$$|x|^{\beta}_{\mathcal{B}} \notin \varnothing$$
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$$|P^1 x|^{\beta}_{\mathcal{B}} = \text{F}$$
$$|\neg P^1 x|^{\beta}_{\mathcal{B}} = \text{T}$$

and similarly:

$$|x|^{\beta}_{\mathcal{B}} \in \{\text{Oxford}\}$$
$$|x|^{\beta}_{\mathcal{B}} \in |Q^1|_{\mathcal{B}}$$
$$|Q^1 x|^{\beta}_{\mathcal{B}} = \text{T}$$
$$|\neg Q^1 x|^{\beta}_{\mathcal{B}} = \text{F}$$

I still need to show that $\forall x\,(\neg P^1x \to \neg Q^1x)$ is false in $\mathcal{B}$. Let β be a variable assignment over $\mathcal{B}$. Then $|x|^\beta_\mathcal{B} = \text{Oxford}$.

$$|x|^\beta_\mathcal{B} \notin \varnothing$$
$$|x|^\beta_\mathcal{B} \notin |P^1|_\mathcal{B}$$
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and similarly:

$$|x|^\beta_\mathcal{B} \in \{\text{Oxford}\}$$
$$|x|^\beta_\mathcal{B} \in |Q^1|_\mathcal{B}$$
$$|Q^1x|^\beta_\mathcal{B} = \text{T}$$
$$|\neg Q^1x|^\beta_\mathcal{B} = \text{F}$$

So I have $|(\neg P^1x \to \neg Q^1x)|^\beta_\mathcal{B} = \text{F}$ and therefore

$$|\forall x\,(\neg P^1x \to \neg Q^1x)|_\mathcal{B} = \text{F}$$

So the conclusion is false in $\mathcal{B}$.