

INTRODUCTION TO LOGIC

4 The Syntax of Predicate Logic

Volker Halbach

I counsel you, dear friend, in sum,
That first you take collegium logicum.
Your spirit's then well broken in for you,
In Spanish boots laced tightly to,
That you henceforth may more deliberately keep
The path of thought and straight along it creep,
And not perchance criss-cross may go,
A- will-o'-wiping to and fro.
Then you'll be taught full many a day
What at one stroke you've done alway,
Like eating and like drinking free,
It now must go like: One! Two! Three!

Goethe, Faust I

4.1 Predicates and Quantification

Some sentences can be parsed into **designators** and **predicate** expressions:

John is tall .
designator predicate

London is bigger than the capital of France .
designator predicate designator

Reginald opens the file with the dvi viewer .
designator first part of predicate designator second part of predicate designator

The argument

Zeno is a tortoise. All tortoises are toothless. Therefore Zeno is toothless.

is logically valid but not propositionally valid: replacing 'Zeno is tortoise', 'All tortoises are toothless', and 'Zeno is toothless' (uniformly) with other sentences doesn't always yield another valid argument.

But the validity is independent of the meaning of 'Zeno', 'tortoise', and 'toothless'; so in order to capture the validity of this argument, I need to analyse the constituents of the sentences.

In the language \mathcal{L}_2 of **predicate logic** such arguments can be analysed.

4.1 Predicates and Quantification

In predicate logic predicate expressions are translated into predicate letters, such as P^2 , Q^1 , R^5 .

The upper index is called the '**arity index**'. It indicates how many designators the predicate takes. In the above examples

'is tall' takes one

'is bigger than' takes two

'opens ... with' takes three

So the predicate expression 'is tall' can be translated as P^1 , 'is bigger than' as Q^2 , and 'opens ... with' as R^3 .

Designators come in different varieties: as **proper names** like ‘Barack Obama’ or ‘the Eiffel Tower’ or as **definite descriptions** like ‘the tallest student in Oxford’; and there are more.

Designators (purport to) refer to one single object.

Especially proper names are formalised in \mathcal{L}_2 as ‘constants’. Constants are a, b, c, a_1, b_1, c_1 , and so on.

Example

Tom hates Mary.

formalisation: P^2ab

dictionary:

P^2 : ... hates ...

a : Tom

b : Mary

The arity index 2 is important: P^2 takes two constants (as ‘hates’ takes two designators).

The order of a and b matters.

40

Using the same dictionary

Example

Tom hates Mary or Mary hates Tom.

is formalised as $(P^2ab \vee P^2ba)$.

Sentences of \mathcal{L}_2 can be combined using connectives in the same way as \mathcal{L}_1 -sentences.

Pronouns

I distinguish two uses of pronouns (like ‘him’, ‘she’ etc.):

(i) lazy uses:

Example

Tom hates Mary and sheMary hates him.Tom.

The pronoun can be replaced by the designator to which it refers back.

(ii) quantificational uses:

Example

A person is morally responsible if and only if she acts freely.a person acts freely.

The pronoun cannot be replaced with the noun to which it refers back (without changing the meaning).

In

Example

A person is morally responsible if and only if she acts freely.

the pronoun ‘she’ is used to express generalisation.

There are other ways to express generalisation, but pronouns offer a very flexible and efficient way of generalising.

Here is how to express a generalisation using only pronouns and generalisations over *all* objects.

Example

All tortoises are reptiles.

This generalisation can be reexpressed using ‘something’: ... or as a generalisation over everything: Replacing pronouns with variables gives the logical form:

If something is a tortoise then it is a reptile.

For everything: if it is a tortoise then it is a reptile.

For all x : if x is a tortoise then x is a reptile

So I need an expression in \mathcal{L}_2 that corresponds to ‘for all’. The symbol \forall is used for that purpose.

Example

If an object₁ is part of another object₂ and it₂ is part of still another object₃, then it₁ is a part of it₃.

Using numerical subscripts one can make the reference of the pronouns clear and unambiguous. In the language \mathcal{L}_2 the variables $x, y, z, x_1, y_1, z_1, x_2, \dots$ play the role of pronouns that are used for quantification.

If an object x_1 is part of another object x_2 and x_2 is part of still another object x_3 , then x_1 is a part of x_3 .

To save on indices I’ll use $x, y, z, x_1, y_1, z_1, x_2, \dots$

I didn’t specify the syntax of \mathcal{L}_2 , and I didn’t say anything about the semantics of \mathcal{L}_2 ; thus we cannot really discuss translations from English into \mathcal{L}_2 . But I’ll sketch how we’ll carry out formalisations in \mathcal{L}_2 .

For formalisations one can again first give the logical form and then replace the English expressions by the corresponding \mathcal{L}_2 -symbols.

Example

All epistemologists are philosophers. For everything: if it is an epistemologist then it is a philosopher. For all x : if x is an epistemologist then x is a philosopher. For all x : (if x is an epistemologist then x is a philosopher) For all x : (if (x is an epistemologist) then x is a philosopher) For all x : (if (x is an epistemologist) then (x is a philosopher))

This is the original sentence. I reexpress the general claim using a pronoun. I replace the pronoun with a variable. 'For all x ' is in logical form. I turn to the remaining sentence 'if x is an epistemologist then x is a philosopher' and apply the methods from propositional logic. As 'if ... then' is a standard connective I put the expression in brackets and turn to the subsentences. ' x is an epistemologist' is a designator and a predicate: it cannot be sensibly be reformulated with a connective or a generalising expression

Example

Some philosophers are logicians.

At least one philosopher is a logician.

There is at least one thing such that it is a philosopher and it is a logician. There is at least one x : x is a philosopher and x is a logician.

There is at least one x : (x is a philosopher and x is a logician)

There is at least one x : ((x is a philosopher) and (x is a logician))

This is the original sentence.

I understand the sentence as saying this.

I reexpress the claim using a pronoun. Now this isn't a generalisation but rather an existential claim. So I put 'at least one thing' rather than 'everything'. 'There is at least one x ' is in logical form.

From the logical form the \mathcal{L}_2 -sentence can be obtained by the following substitutions

Example

For all x : (if (x is an epistemologist) then (x is a philosopher))

This is the logical form. The standard connectives are replaced by the respective symbols. 'for all' is replaced by \forall . ' x is an epistemologist' is formalised as the atomic formula Q^1x , and ' x is a philosopher' is formalised as P^1x . as in the case of propositional logic the brackets around sentence that are not further analysable are dropped. So the sentence is formalised as $\forall x (Q^1x \rightarrow P^1x)$ with the following dictionary:

P^1 : ... is a philosopher

Q^1 : ... is an epistemologist

Example

There is at least one x : ((x is a philosopher) and (x is a logician))

This is formalised as:

$\exists x (P^1x \wedge R^1x)$

P^1 : ... is a philosopher

R^1 : ... is a logician

Example

All persons have a soul.

For all x : if x is a person then x has a soul

For all x : (if x is a person then x has a soul)

For all x : (if x is a person then x has a soul)

For all x : (if (x is a person) then x has a soul)

For all x : (if (x is a person) then x has a soul)

For all x : (if ((x is a person) then there is at least one y : x has y and y is a soul) For all x : (if (x is a person) then there is at least one y : (x has y and y is a soul)) For all x : (if (x is a person) then there is at least one y : ((x has y) and (y is a soul)))

4.2 The Sentences of \mathcal{L}_2

Everything up to this point is just an informal blurb motivating the following definitions.

I am now going to specify the syntax of \mathcal{L}_2 in precise terms.

Definition (predicate letters)

All expressions of the form P_n^k , Q_n^k , or R_n^k are **predicate letters**, where k and n are either missing (no symbol) or a numeral '1', '2', '3', ...

So the letter P with or without numerals '1', '2', and so on as upper and/or lower indices is a predicate letter, and similarly for Q and R . The sentence letters P , Q , R , P_1 , Q_1 , ... are also predicate letters, according to this definition. Furthermore, P^1 , Q^1 , R^1 , P_1^1 , Q_1^1 , R_1^1 , P_2^1 , Q_2^1 , R_2^1 , ..., P_1^2 , Q_1^2 , R_1^2 , P_2^2 , Q_2^2 , R_2^2 , and so on, are predicate letters.

20

Starting from the logical I go on to the formalisation:

Example

For all x : (if (x is a person) then there is at least one y : ((x has y) and (y is a soul))) $\forall x ((x \text{ is a person}) \rightarrow \exists y ((x \text{ has } y) \wedge (y \text{ is a soul})))$

$\forall x (P^1x \rightarrow \exists y (Q^2xy \wedge R^1y))$

This is the logical form.

I introduce the symbols for connectives and quantifiers.

Predicate expressions are replaced by predicate letters of an appropriate arity... .. using the following dictionary:

P^1 is a person

4.2 The Sentences of \mathcal{L}_2

Definition

The value of the upper index of a predicate letter is called its **arity**. If a predicate letter does not have an upper index its arity is 0.

Definition (constants)

a , b , c , a_1 , b_1 , c_1 , a_2 , b_2 , c_2 , a_3 , ... are **constants**.

Definition (variables)

x , y , z , x_1 , y_1 , z_1 , x_2 , ... are **variables**.

Definition (atomic formulae of \mathcal{L}_2)

If Z is a predicate letter of arity n and each of t_1, \dots, t_n is a variable or a constant, then $Zt_1 \dots t_n$ is an **atomic formula** of \mathcal{L}_2 .

Example

The following expressions are atomic formulae of \mathcal{L}_2 :

$$Q^1x$$

$$P^2cy$$

$$P^3_{531}c_4y$$

$$R^2xx$$

Definition

A quantifier is an expression $\forall v$ or $\exists v$ where v is a variable.

Thus, $\forall x_{348}$ and $\exists z$ are quantifiers.

Definition (formulae of \mathcal{L}_2)

- (i) All atomic formulae of \mathcal{L}_2 are formulae of \mathcal{L}_2 .
- (ii) If ϕ and ψ are formulae of \mathcal{L}_2 , then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$ are formulae of \mathcal{L}_2 .
- (iii) If v is a variable and ϕ is a formula, then $\forall v \phi$ and $\exists v \phi$ are formulae of \mathcal{L}_2 .

Example

The following expressions are formulae of \mathcal{L}_2 :

$$\forall x (P^2xa \rightarrow Q^1x)$$

$$\forall z_{77} \neg \exists y_3 \exists z_{45} (P^2xy \rightarrow \exists x_2 (R^4_{377}c_3xz_{77} \wedge Q))$$

$$(\exists x P^1x \leftrightarrow \neg \exists y \exists y Q^2yy)$$

$$\forall x \exists z R^2az$$

The formula P^1x isn't a sentence. Only once the variable x is used or *bound* by some quantifier is becomes a sentence.

Roughly speaking, an occurrence of a variable is bound iff it refers back to a quantifier; otherwise the occurrence is free.

Example

$$\forall x (P^1x \rightarrow Q^2ax)$$

In this formula both occurrences of the variable x are free. Now both occurrences of the variable x refer back to the quantifier $\forall x$, so they are both **bound**.

Example

$$(\forall x P^1x \rightarrow Q^2ax)$$

In this formula only the first red occurrence of x refers back to $\forall x$; it's bound by this quantifier; the second (i.e. green) occurrence is free.

Definition

- (i) All occurrences of variables in atomic formulae are free.
- (ii) The occurrences of a variable that are free in ϕ and ψ are also free in $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$.
- (iii) In a formula $\forall v \phi$ no occurrence of the variable v is free; all occurrences of variables other than v that are free in ϕ are also free in $\forall v \phi$.
- (iv) In a formula $\exists v \phi$ no occurrence of the variable v is free; all occurrences of variables other than v that are free in ϕ are also free in $\exists v \phi$.

An occurrence of a variable is bound in a formula if and only if it is not free.

I look at my example again to illustrate the definition:

Example

$$P^1x \quad \forall x P^1x \quad \forall x P^1x \quad Q^2ax$$

$$(\forall x P^1x \rightarrow Q^2ax)$$

P^1x is an atomic formula... Writing $\forall x$ in front of P^1x binds the green occurrence of x . Q^2ax is an atomic formula, so the red occurrence of x is free. Q^2ax is still not related to P^1x . Now I combine the two formulae using \rightarrow but that doesn't make the red occurrence of x a bound occurrence.

Definition

A variable **occurs freely** in a formula if and only if there is at least one free occurrence of the variable in the formula.

Definition (sentence of \mathcal{L}_2)

A formula of \mathcal{L}_2 is a **sentence** of \mathcal{L}_2 if and only if no variable occurs freely in the formula.

This section doesn't concern the syntax of \mathcal{L}_2 ; it just contains some rules for abbreviating formulae of \mathcal{L}_2 . These rules do not form part of the syntax of \mathcal{L}_2 , they just are conventions that allow one to abbreviate formulae.

The bracketing conventions of \mathcal{L}_1 apply also to \mathcal{L}_2 formulae.

Convention

An \mathcal{L}_2 -formula may be abbreviated by dropping the arity indices.

So instead of P^2xy one may write Pxy .

Example

$$\forall x (((P^1x \wedge R_{\frac{2}{5}}^2xa)) \rightarrow \exists y_2(((R_{\frac{2}{5}}^2xy_2 \wedge Q^1x)) \wedge P^1y_2))$$

This is the sentence. Now I apply the rules for abbreviating formulae of \mathcal{L}_2 . The connective \wedge binds more strongly than \rightarrow ; so I drop the red brackets. The pair of green brackets can be dropped because in chains of formulae with \wedge 'left-bracketing' applies. And then, according to the new rule, I can drop the arity indices from all predicate letters. Note that there is no pair of outer brackets that I could drop. The formula cannot be further abbreviated.

In the abbreviation

$$Pa \wedge Pab$$

the two occurrences of P stand for different predicate letters, which becomes obvious when arity indices are added:

$$(P^1a \wedge P^2ab)$$

This is only my first stab on formalisation in \mathcal{L}_2 . In Chapter 7 I'll take up the topic again after specifying the semantics of \mathcal{L}_2 .