INTRODUCTION TO LOGIC

2 Syntax and Semantics of Propositional Logic

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Logic is the beginning of wisdom.
Thomas Aquinas

In what follows I look at some formal languages that are *much* simpler than English and define *validity of arguments*, 'truth under an interpretation', *consistency* etc. for these formal languages.

In logic one abstracts from all stylistic variants etc of natural language and retains just the basic skeleton of the language in a regimented form.

When presenting a formal language, I proceed in the following order:

1. I specify the syntax or grammar of the language; in particular I define what the sentences of the language are.
2. I specify the semantics of the language; in particular, I say what it means for a sentence to be true under an interpretation (or in a 'structure'). Once the notion of an interpretation (or structure) is clear, I can define validity of arguments etc as for English.

Syntax is all about *expressions*: words and sentences.

Examples of syntactic claims

- 'Bertrand Russell' is a proper noun.
- 'likes logic' is a verb phrase.
- 'Bertrand Russell likes logic' is a sentence.
- Combining a proper noun and a verb phrase in this way yields a sentence.

Semantics is all about *meanings* of expressions.

Examples of semantic claims

- 'Bertrand Russell' refers to a British philosopher.
- 'Bertrand Russell' refers to Bertrand Russell.
- 'likes logic' expresses a property Russell has.
- 'Bertrand Russell likes logic' is true.

Note our use of quotes to talk about expressions.

'Bertrand Russell' refers to Bertrand Russell.

Mention

- The first occurrence of 'Bertrand Russell' is an example of mention.
- This occurrence (with quotes) refers to an expression.

Use

- The second occurrence of 'Bertrand Russell' is an example of use.
- This occurrence (without quotes) refers to a man.

Syntax: English vs. $\mathcal{L}_1$.

English has *many* different sorts of expressions.

Some expressions of English

(1) *Sentences*: 'Bertrand Russell likes logic', 'Philosophers like conceptual analysis', etc..

(2) *Connectives*: 'it is not the case that', 'and', etc..

(3) *Noun phrases*: 'Bertrand Russell', 'Philosophers', etc..

(4) *Verb phrases*: 'likes logic', 'like conceptual analysis', etc..

(5) Also: *nouns*, *verbs*, *pronouns*, etc., etc., etc..

$\mathcal{L}_1$ has *just two* sorts of basic expressions.

Some basic expressions of $\mathcal{L}_1$

(1) *Sentence letters*: e.g. 'P', 'Q'.

(2) *Connectives*: e.g. '¬', '∧'. There are also brackets: '(' and ')'.

Combining sentences and connectives makes new sentences.

Some complex sentences

- 'It is not the case that' and 'Bertrand Russell likes logic' make:

  'It is not the case that Bertrand Russell likes logic'.
- '¬' and 'P' make: '$\neg P$'.
- 'Bertrand Russell likes logic' and 'and' and 'Philosophers like conceptual analysis' make:

  'Bertrand Russell likes logic and philosophers like conceptual analysis'.
- 'P', '∧' and 'Q' make: '$(P \wedge Q)$'.

Logic convention: no quotes around $\mathcal{L}_1$-expressions.

- P, $\wedge$ and Q make: $(P \wedge Q)$.

Connectives

Here's the full list of $\mathcal{L}_1$-connectives.

name	in English	symbol
conjunction	and	$\wedge$
disjunction	or	$\vee$
negation	it is not the case that	$\neg$
arrow	if … then	$\rightarrow$
double arrow	if and only if	$\leftrightarrow$

The syntax of $\mathcal{L}_1$

Here's the official definition of $\mathcal{L}_1$-sentence.

Definition

(i) All sentence letters are sentences of $\mathcal{L}_1$:
- $P, Q, R, P_1, Q_1, R_1, P_2, Q_2, R_2, P_3, \ldots$

(ii) If ϕ and ψ are sentences of $\mathcal{L}_1$, then so are:
- $\neg\phi$
- $(\phi \wedge \psi)$
- $(\phi \vee \psi)$
- $(\phi \rightarrow \psi)$
- $(\phi \leftrightarrow \psi)$

(iii) Nothing else is a sentence of $\mathcal{L}_1$.

Greek letters: ϕ ('PHI') and ψ ('PSI'): not part of $\mathcal{L}_1$.

How to build a sentence of $\mathcal{L}_1$

Example

The following is a sentence of $\mathcal{L}_1$:

$$\neg\neg(((P \wedge Q) \rightarrow (P \vee \neg R_{45})) \leftrightarrow \neg((P_3 \vee R) \vee R))$$

Definition of $\mathcal{L}_1$-sentences (repeated from previous page)

(i) All sentence letters are sentences of $\mathcal{L}_1$.

(ii) If ϕ and ψ are sentences of $\mathcal{L}_1$, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$ and $(\phi \leftrightarrow \psi)$ are sentences of $\mathcal{L}_1$.

(iii) Nothing else is a sentence of $\mathcal{L}_1$.

I mentioned that ϕ and ψ are *not* part of $\mathcal{L}_1$.

- $\neg P$ is a $\mathcal{L}_1$-sentence.
- $\neg\phi$ describes many $\mathcal{L}_1$-sentences (but is not one itself).
  e.g. $\neg P$, $\neg(Q \vee R)$, $\neg(P \leftrightarrow (Q \vee R))$, …

ϕ and ψ are part of the metalanguage, not the object one.

Object language

The object language is the one we are theorising *about*.
- The object language is $\mathcal{L}_1$.

Metalanguage

The metalanguage is the one we are theorising *in*.
- The metalanguage is (augmented) English.

ϕ and ψ are used as variables in the metalanguage:
in order to generalise about sentences of the object language.

Bracketing conventions

There are conventions for dropping brackets in $\mathcal{L}_1$.
Some are similar to rules used for + and × in arithmetic.

Example in arithmetic

- $4 + 5 \times 3$ does not abbreviate $(4 + 5) \times 3$.
- × 'binds more strongly' than +.
  $4 + 5 \times 3$ abbreviates $4 + (5 \times 3)$.

Conventions in $\mathcal{L}_1$

- $\wedge$ and $\vee$ bind more strongly than $\to$ and $\leftrightarrow$.
  $(P \to Q \wedge R)$ abbreviates $(P \to (Q \wedge R))$.
- One may drop outer brackets.
  $P \wedge (Q \to \neg P_4)$ abbreviates $(P \wedge (Q \to \neg P_4))$.
- One may drop brackets on strings of $\wedge$s or $\vee$s that are bracketed to the left.
  $(P \wedge Q \wedge R)$ abbreviates $((P \wedge Q) \wedge R)$.

Semantics

Recall the characterisation of validity from week 1.

Characterisation

An argument is *logically valid* if and only if there is *no* interpretation of subject-specific expressions under which:

(i) the premisses are all true, and
(ii) the conclusion is false.

We'll adapt this characterisation to $\mathcal{L}_1$.

- Logical expressions: $\neg, \wedge, \vee, \to$ and $\leftrightarrow$.
- Subject-specific expressions: $P, Q, R, \ldots$
- Interpretation: $\mathcal{L}_1$-*structure*.

$\mathcal{L}_1$-structures

We interpret sentence letters by assigning them truth-values: either T for True or F for False.

Definition

An $\mathcal{L}_1$-*structure* is an assignment of exactly one truth-value (T or F) to every sentence letter of $\mathcal{L}_1$.

Examples

We can think of an $\mathcal{L}_1$-structure as an infinite list that provides a value T or F for every sentence letter.

	P	Q	R	P_1	Q_1	R_1	P_2	Q_2	R_2	$\cdots$
$\mathcal{A}$:	T	F	F	F	T	F	T	T	F	$\cdots$
$\mathcal{B}$:	F	F	F	F	F	F	F	F	F	$\cdots$

We use $\mathcal{A}$, $\mathcal{B}$, etc. to stand for $\mathcal{L}_1$-structures.

Truth-values of complex sentences 1/3

$\mathcal{L}_1$-structures *only* directly specify truth-values for $P, Q, R, \ldots$

- The logical connectives have fixed meanings.
- These determine the truth-values of complex sentences.
- Notation: $|\phi|_{\mathcal{A}}$ is the truth-value of ϕ under $\mathcal{A}$.
- For all $\mathcal{L}_1$-structures $\mathcal{A}$ and sentences ϕ we have either $|\phi|_{\mathcal{A}} = T$ or $|\phi|_{\mathcal{A}} = F$.

Truth-conditions for $\neg$

The meaning of $\neg$ is summarised in its *truth table*.

ϕ	$\neg\phi$
T	F
F	T

In words: $|\neg\phi|_{\mathcal{A}} =$ T if and only if $|\phi|_{\mathcal{A}} =$ F.

Worked example 1

$|\phi|_{\mathcal{A}}$ is the truth-value of ϕ under $\mathcal{A}$.

ϕ	$\neg\phi$
T	F
F	T

Compute the following truth-values.

Let the structure $\mathcal{A}$ be partially specified as follows.

P	Q	R	P_1	Q_1	R_1	P_2	Q_2	R_2	$\cdots$
T	F	F	F	T	F	T	T	F	$\cdots$

Compute:

$\lvert P\rvert_{\mathcal{A}} =$ T	$\lvert Q\rvert_{\mathcal{A}} =$ F	$\lvert R_1\rvert_{\mathcal{A}} =$ F
$\lvert\neg P\rvert_{\mathcal{A}} =$ F	$\lvert\neg Q\rvert_{\mathcal{A}} =$ T	$\lvert\neg R_1\rvert_{\mathcal{A}} =$ T
$\lvert\neg\neg P\rvert_{\mathcal{A}} =$ T	$\lvert\neg\neg Q\rvert_{\mathcal{A}} =$ F	$\lvert\neg\neg R_1\rvert_{\mathcal{A}} =$ F

Truth-values of complex sentences 2/3

Truth-conditions for $\wedge$ and $\vee$

The meanings of $\wedge$ and $\vee$ are given by the truth tables:

ϕ	ψ	$(\phi \wedge \psi)$
T	T	T
T	F	F
F	T	F
F	F	F

ϕ	ψ	$(\phi \vee \psi)$
T	T	T
T	F	T
F	T	T
F	F	F

$|(\phi \wedge \psi)|_{\mathcal{A}} =$ T if and only if $|\phi|_{\mathcal{A}} =$ T and $|\psi|_{\mathcal{A}} =$ T.
$|(\phi \vee \psi)|_{\mathcal{A}} =$ T if and only if $|\phi|_{\mathcal{A}} =$ T or $|\psi|_{\mathcal{A}} =$ T (or both).

Truth-values of complex sentences 3/3

Truth-conditions for $\rightarrow$ and $\leftrightarrow$

The meanings of $\rightarrow$ and $\leftrightarrow$ are given by the truth tables:

ϕ	ψ	$(\phi \rightarrow \psi)$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	ψ	$(\phi \leftrightarrow \psi)$
T	T	T
T	F	F
F	T	F
F	F	T

$|(\phi \rightarrow \psi)|_{\mathcal{A}} =$ T if and only if $|\phi|_{\mathcal{A}} =$ F or $|\psi|_{\mathcal{A}} =$ T.
$|(\phi \leftrightarrow \psi)|_{\mathcal{A}} =$ T if and only if $|\phi|_{\mathcal{A}} = |\psi|_{\mathcal{A}}$.

Worked example 2

Let $|P|_{\mathcal{B}} =$ T and $|Q|_{\mathcal{B}} =$ F.

Compute $|\neg(P \rightarrow Q) \rightarrow (P \wedge Q)|_{\mathcal{B}}$

What is the truth value of $\neg(P \rightarrow Q) \rightarrow (P \wedge Q)$ under $\mathcal{B}$?

1. $|(P \rightarrow Q)|_{\mathcal{B}} =$ F and $|(P \wedge Q)|_{\mathcal{B}} =$ F
2. $|\neg(P \rightarrow Q)|_{\mathcal{B}} =$ T
3. $|\neg(P \rightarrow Q) \rightarrow (P \wedge Q)|_{\mathcal{B}} =$ F

ϕ	$\neg\phi$
T	F
F	T

ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \rightarrow \psi)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

For actual calculations it's usually better to use tables.

Suppose $|P|_{\mathcal{B}} = \mathrm{T}$ and $|Q|_{\mathcal{B}} = \mathrm{F}$.

Compute $|\neg(P \to Q) \to (P \wedge Q)|_{\mathcal{B}}$

P	Q	$\neg$	$(P$	$\to$	$Q)$	$\to$	$(P$	$\wedge$	$Q)$
T	F	T	T	F	F	**F**	T	F	F

ϕ	$\neg\phi$
T	F
F	T

ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \to \psi)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

Using the same technique we can fill in the full truth table for $\neg(P \to Q) \to (P \wedge Q)$

P	Q	$\neg$	$(P$	$\to$	$Q)$	$\to$	$(P$	$\wedge$	$Q)$
T	T	F	T	T	T	**T**	T	T	T
T	F	T	T	F	F	**F**	T	F	F
F	T	F	F	T	T	**T**	F	F	T
F	F	F	F	T	F	**T**	F	F	F

The main column (in boldface) gives the truth-value of the whole sentence.

ϕ	$\neg\phi$
T	F
F	T

ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \to \psi)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

Validity

Let Γ be a set of sentences of $\mathcal{L}_1$ and ϕ a sentence of $\mathcal{L}_1$.

Definition

The argument with all sentences in Γ as premisses and ϕ as conclusion is *valid* if and only if there is no $\mathcal{L}_1$-structure under which:

(i) all sentences in Γ are true; and

(ii) ϕ is false.

Notation: when this argument is valid we write $\Gamma \vDash \phi$.

$\{P \to \neg Q, Q\} \vDash \neg P$ means that the argument whose premises are $P \to \neg Q$ and Q, and whose conclusion is $\neg P$ is valid.
Also written: $P \to \neg Q, Q \vDash \neg P$

Worked example 3

We can use truth-tables to show that $\mathcal{L}_1$-arguments are valid.

Example

Show that $\{P \to \neg Q, Q\} \vDash \neg P$.

	P	Q	P	$\to$	$\neg$	Q	Q	$\neg$	P
►	T	T	T	**F**	F	T	**T**	**F**	T
►	T	F	T	**T**	T	F	**F**	**F**	T
►	F	T	F	**T**	F	T	**T**	**T**	F
►	F	F	F	**T**	T	F	**F**	**T**	F

Rows correspond to interpretations.
One needs to check that there is no row in which all the premisses are assigned T and the conclusion is assigned F.

Other logical notions

Definition

A sentence ϕ of $\mathcal{L}_1$ is *logically true* (a *tautology*) iff:

- ϕ is true under all $\mathcal{L}_1$-structures.

e.g. $P \lor \neg P$, and $P \to P$ are tautologies.

Truth tables of tautologies

Every row in the main column is a T.

P	P	$\lor$	$\neg$	P	P	$\to$	P
T	T	**T**	F	T	T	**T**	T
F	F	**T**	T	F	F	**T**	F

Definition

A sentence ϕ of $\mathcal{L}_1$ is a *contradiction* iff:

- ϕ is not true under any $\mathcal{L}_1$-structure.

e.g. $P \land \neg P$, and $\neg(P \to P)$ are contradictions.

Truth tables of contradictions

Every row in the main column is an F.

P	P	$\land$	$\neg$	P	$\neg$	$(P$	$\to$	$P)$
T	T	**F**	F	T	**F**	T	T	T
F	F	**F**	T	F	**F**	F	T	F

Definition

Sentences ϕ and ψ are *logically equivalent* iff:

- ϕ and ψ are true in exactly the same $\mathcal{L}_1$-structures.

- P and $\neg\neg P$ are logically equivalent.
- $P \land Q$ and $\neg(\neg P \lor \neg Q)$ are logically equivalent.

Truth tables of logical equivalents

The truth-values in the main columns agree.

P	Q	P	$\land$	Q	$\neg$	$(\neg$	P	$\lor$	$\neg$	$Q)$
T	T	T	**T**	T	**T**	F	T	F	F	T
T	F	T	**F**	F	**F**	F	T	T	T	F
F	T	F	**F**	T	**F**	T	F	T	F	T
F	F	F	**F**	F	**F**	T	F	T	T	F

Worked example 4

Example

Show that the sentence $(P \to (\neg Q \land R)) \lor P$ is a tautology.

Method 1: Full truth table

- Write out the truth table for $(P \to (\neg Q \land R)) \lor P$.
- Check there's a T in every row of the main column.

P	Q	R	$(P$	$\to$	$(\neg$	Q	$\land$	$R))$	$\lor$	P
T	T	T	T	F	F	T	F	T	**T**	T
T	T	F	T	F	F	T	F	F	**T**	T
T	F	T	T	T	T	F	T	T	**T**	T
T	F	F	T	F	T	F	F	F	**T**	T
F	T	T	F	T	F	T	F	T	**T**	F
F	T	F	F	T	F	T	F	F	**T**	F
F	F	T	F	T	T	F	T	T	**T**	F
F	F	F	F	T	T	F	F	F	**T**	F

Worked example 4 (cont.)

Show that the sentence $(P \to (\neg Q \wedge R)) \vee P$ is a tautology.

Method 2: Backwards truth table.

- Put an F in the main column.
- Work backwards to show this leads to a contradiction.

P	Q	R	$(P$	$\to$	$(\neg$	Q	$\wedge$	$R))$	$\vee$	P
			T_3?	F_1					F	F_2

ϕ	$\neg\phi$
T	F
F	T

ϕ	ψ	$(\phi \wedge \psi)$	$(\phi \vee \psi)$	$(\phi \to \psi)$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T