The solutions are highly incomplete and only intended to give a rough idea.

1. (a) Which of the following expressions is an abbreviation of a sentence of $\mathscr{L}_1$? If an expression is an abbreviation of a $\mathscr{L}_1$-sentence, then restore all the brackets that have been dropped in accordance with the Bracketing Conventions of the Logic Manual. If not, explain why not. [4]

   (i) $P \vee Q \vee R \leftrightarrow \neg(\neg P \wedge \neg Q \wedge \neg R)$

   This expression is an abbreviation of the following $\mathscr{L}_1$-sentence:

$$(((P \vee Q) \vee R) \leftrightarrow \neg((\neg P \wedge \neg Q) \wedge \neg R))$$

   (ii) $P \vee Q \wedge R \rightarrow P \vee (Q \rightarrow R)$

   This expression is not *an abbreviation of a $\mathscr{L}_1$-sentence because $P \vee Q \wedge R$ is not an abbreviation of one. None of the bracketing conventions allow us to obtain that expression after dropping any brackets from a $\mathscr{L}_1$-sentence.*

   [1] (b) What is it for two $\mathscr{L}_1$-sentences to be logically equivalent?

   (c) Which of the following are pairs of logically equivalent $\mathscr{L}_1$-sentences? Use truth tables in order to justify your answer. [4]

   (i) $(P \rightarrow Q) \wedge \neg Q$, $P \rightarrow Q \wedge \neg Q$

   (ii) $P \rightarrow (Q \rightarrow R)$, $R \rightarrow (\neg Q \rightarrow \neg P)$

   (d) Show that the following argument can be turned into a propositionally valid argument by appropriately rewording its premises and adding any further assumptions upon which the speaker might naturally be expected to be relying if required. Justify your answer by means of a partial truth table or a proof in Natural Deduction. Specify your dictionary carefully and note any difficulties or points of interest. [16]

   I just raised my hand. Either determinism is false or the laws of nature being what they in fact are and the events in the remote past being what they in fact are entailed my raising my hand just now. But if the laws of nature being what they are together with the events in the remote past being what they are entailed my raising my hand, then I only had any control over my doing so if I had any control over the laws of nature being what they in fact are or the events in the remote past being what they in fact are. It is plain that I had no control whatever over the laws of nature being what they in fact are. Nor did I had any control over the events in the remote past being what they in fact are. It follows that I had no control whatever over my raising my hand. For determinism is true after all.

Answer:

Consider the following dictionary:

P: Determinism is true
Q: I raise my hand
R_1: The laws of nature are what they in fact are
R_2: The events in the remote past are what they in fact are
Q_1: I had control over my raising my hand
R_3: I had control over the laws of nature being what they are
R_4: I had control over the events in the remote past being what they are

The following is a formalisation of the argument:

Q
$\neg P \vee (R_1 \wedge R_2 \to Q)$
$(R_1 \wedge R_2 \to Q) \to (Q_1 \to R_3 \vee R_4)$
$\neg R_3$
$\neg R_4$
P
$\neg Q_1$

Some points of interest:

- *We translated 'It is plain that we had no control over the laws of nature being what they in fact are' as 'I had no control over the laws of nature being what they in fact are.'*

- ...

Q	$\neg P \vee (R_1 \wedge R_2 \to Q)$	$(R_1 \wedge R_2 \to Q) \to (Q_1 \to R_3 \vee R_4)$	$\neg R_3$	$\neg R_4$	P	$\neg Q_1$
T	FT T T	T T T T F T ?	T F	T F	T	FT

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P \vee (R_1 \wedge R_2 \to Q) \qquad \dfrac{[\neg P] \qquad P}{R_1 \wedge R_2 \to Q} \qquad [R_1 \wedge R_2 \to Q]}{R_1 \wedge R_2 \to Q} \qquad (R_1 \wedge R_2 \to Q) \to (Q_1 \to R_3 \vee R_4)}{Q_1 \to R_3 \vee R_4} \qquad [Q_1]}{R_3 \vee R_4} \qquad \dfrac{[R_3] \quad \neg R_3}{\neg Q_1} \qquad \dfrac{[R_4] \quad \neg R_4}{\neg Q_1}}{\neg Q_1} \qquad [Q_1]}{\neg Q_1}
$$

2. (a) Show that each of the following arguments is in fact propositionally valid. You may use a partial truth table or a Natural Deduction proof to show that
[9] an argument is propositionally valid.

   (i) If Brown decides to visit Spain, then he will not visit Barcelona. Therefore, if Brown visits Barcelona, then Brown will not decide to visit Spain.

   $P \to \neg Q \vdash Q \to \neg P$

$$\frac{\dfrac{\dfrac{P \to \neg Q \qquad [P]}{\neg Q} \qquad [Q]}{\neg P}}{Q \to \neg P}$$

   (ii) It is not the case that, if God exists, then human life is the product of random chance. So, God exists.

   $\neg(P \to Q) \vdash P$

$$\frac{\neg(P \to Q) \qquad \dfrac{\dfrac{[\neg P] \qquad [P]}{Q}}{P \to Q}}{P}$$

   (iii) If I turn the ignition on, then the engine will start. Therefore, if I turn the ignition on and there is no petrol in the tank, then the engine will start.

   $P \to R \vdash P \wedge \neg Q \to R$

$$\frac{\dfrac{\dfrac{[P \wedge \neg Q]}{P} \qquad P \to R}{R}}{P \to R}$$

(b) Write a brief essay discussing the question of whether we should conclude that $\to$ is inadequate for the formalisation of 'if ... then ...' in English or whether we could still argue that, despite appearances to the contrary, $\to$ provides an appropriate translation for the use of 'if ... then ...' in the
[16] previous three arguments.

3. (a) Write an essay comparing and contrasting the following two rival characterisations of logical validity. Are there any reasons to prefer one
[13] over the other?

   (A) *An argument is logically valid if and only if there is no interpretation under which the premises are true and the conclusion is false.*

(B) *An argument is logically valid if and only if it is impossible for the premises to be true while the conclusion is false.*

(b) Which of the following arguments are logically valid? If an argument has a valid formalisation in $\mathscr{L}_1$, $\mathscr{L}_2$ or $\mathscr{L}_=$, then specify your dictionary and use a partial truth table in the case of arguments in $\mathscr{L}_1$ or a Natural Deduction proof in the case of arguments in $\mathscr{L}_2$ or $\mathscr{L}_=$ to show the validity of the resulting argument. If an argument cannot be formalised into a valid argument in $\mathscr{L}_=$, then specify your dictionary and provide a counterexample. (There is no need to prove that your structure is a counterexample; you need only specify the structure and briefly sketch the reasons why it counts as a counterexample.) Finally, explain whether your formalisation is adequate. [12]

(i) Some apples are red all over and some apples are green all over. As a consequence, some apples are both red and green all over.

$\exists x(Px \wedge Q_1 x), \exists x(Px \wedge Q_2 x) \nvDash \exists x(Px \wedge Q_1 x \wedge Q_2 x)$

$$D_{\mathscr{S}} = \{1,2\}$$
$$|P|_{\mathscr{S}} = \{1,2\}$$
$$|Q_1|_{\mathscr{S}} = \{1\}$$
$$|Q_2|_{\mathscr{S}} = \{2\}$$

Since the preceding formalisation is arguably adequate, it follows that the argument is not valid.

(ii) Nothing is both red and green all over.

$\nvDash \neg\exists x(Q_1 x \wedge Q_2 x)$

The preceding counterexample will do. So, its formalisation in $\mathscr{L}_2$ is not valid. Arguably, however, the preceding formalisation is in fact adequate, which means the argument is not valid.

(iii) Most tutors hold their tutorials in the afternoon. Most tutors hold their tutorials in College. Therefore, some tutors must hold their tutorials in College in the afternoon.

$\exists x(Px \wedge Q_1 x), \exists x(Px \wedge Q_2 x) \nvDash \exists x(Px \wedge Q_1 x \wedge Q_2 x)$

$$D_{\mathscr{S}} = \{1,2\}$$
$$|P|_{\mathscr{S}} = \{1,2\}$$
$$|Q_1|_{\mathscr{S}} = \{1\}$$
$$|Q_2|_{\mathscr{S}} = \{2\}$$

Arguably, however, the preceding formalisation of the argument in $\mathscr{L}_2$ is not adequate. Unfortunately, predicate logic lacks the resources to *do justice to the English quantifier 'most' which should presumably count as a topic-neutral expression. So, the argument is valid even if it lacks an adequate formalisation in predicate logic.*

(iv) If God is perfectly good, then there is evil in the world only if God cannot prevent it. If there is evil in the world, and God cannot prevent it, then God is not omnipotent. But there is evil in the world. Therefore, God is not both omnipotent and perfectly good.

$P \to (Q \to \neg R), Q \wedge \neg R \to \neg P_1, Q \vdash \neg(P \wedge P_1)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P\wedge P_1]}{P} \quad P\to(Q\to\neg R)}{Q\to\neg R} \quad Q}{\neg R} \quad Q}{Q\wedge\neg R} \quad Q\wedge\neg R\to\neg P_1}{\neg P_1} \quad \dfrac{[P\wedge P_1]}{P_1}}{\neg(P\wedge P_1)}}{}$$

[12] 4. (a) Establish each of the following by means of a proof in the system of Natural Deduction:

(i) $P_1 \vee P_2 \to Q \vee R, \neg(P_1 \to Q) \vdash P_2 \to R$

$$\dfrac{\dfrac{\dfrac{\dfrac{[P_2]}{P_1\vee P_2}\quad P_1\vee P_2\to Q\vee R}{Q\vee R}\quad \dfrac{\dfrac{[Q]}{P_1\to Q}\quad \neg(P_1\to Q)}{R}\quad [R]}{R}}{P_2\to R}$$

(ii) $\forall x(Px \to (Q_1x \vee Q_2x)), \neg\exists x(Q_1x \wedge Rx), \exists x(Px \wedge Rx) \vdash \exists x Q_2x$

(iii) $\exists x Rxx, \forall x\forall y(Rxx \wedge Rxy \to x = y) \vdash \exists x\forall y(Rxy \leftrightarrow x = y)$

$$\dfrac{\exists xRxx \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Raa]\quad [Rab]}{Raa\wedge Rab}\quad \dfrac{\dfrac{\forall x\forall y(Rxx\wedge Rxy\to x=y)}{\forall y(Raa\wedge Ray\to a=y)}}{Raa\wedge Rab\to a=b}}{a=b}}{Rab\to a=b}\quad \dfrac{\dfrac{[a=b]\quad [Raa]}{Rab}}{a=b\to Rab}}{Rab\leftrightarrow a=b}}{\forall y(Ray\leftrightarrow a=y)}}{\exists x\forall y(Rxy\leftrightarrow x=y)}}{\exists x\forall y(Rxy\leftrightarrow x=y)}$$

$$
\dfrac{\exists x(Px \wedge Rx) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Pa \wedge Ra]}{Pa} \qquad \dfrac{\forall x(Px \rightarrow (Q_1x \vee Q_2x))}{Pa \rightarrow (Q_1a \vee Q_2a)}}{Q_1a \vee Q_2a} \qquad \dfrac{\dfrac{\dfrac{[Q_1a] \qquad \dfrac{Pa \wedge Ra}{Ra}}{Q_1a \wedge Ra}}{\exists x(Q_1x \wedge Rx)} \qquad \neg\exists x(Q_1x \wedge Rx)}{Q_2a} \qquad [Q_2a]}{Q_2a}}{\exists x Q_2x}}{\exists x Q_2x}
$$

(b) Find any mistakes in the following attempted proofs. List all steps in the proof that are not licensed by a rule of Natural Deduction. If there is a repair, supply a correct proof. If not, show that the argument is not valid by means of a counterexample. (There is no need to prove that your structure is a counterexample; you need only specify the structure and briefly sketch the
[13] reasons why it counts as a counterexample.)

(i) $R, \neg(P \land Q \land R), \neg P \lor \neg R \to P_1 \land P_2 \vdash Q \to P_1 \land P_2$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{R \quad [Q]}{Q \land R} \quad [P]}{P \land Q \land R} \quad \neg(P \land Q \land R)}{\neg P}}{\neg P \lor \neg R} \quad \neg P \lor \neg R \to P_1 \land P_2}{P_1 \land P_2}}{Q \to P_1 \land P_2}$$

Answer.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{R \quad \cfrac{[Q] \quad [P]}{P \land Q}}{P \land Q \land R} \quad \neg(P \land Q \land R)}{\neg P}}{\neg P \lor \neg R} \quad \neg P \lor \neg R \to P_1 \land P_2}{P_1 \land P_2}}{Q \to P_1 \land P_2}$$

(ii) $\vdash \neg(\exists x Px \to \forall x Px) \to (\exists x \neg Px \to \exists x Px)$

$$\cfrac{\exists x \neg Px \quad \cfrac{\cfrac{\neg Pa \quad \cfrac{[\forall x Px]}{Pa}}{\neg \forall x Px} \quad \cfrac{\neg \exists x Px \to \forall x Px \quad [\neg \exists x Px]}{\forall x Px}}{\exists x Px}}{\exists x Px}$$

Answer.

$$\cfrac{\cfrac{\neg(\exists x Px \to \forall x Px) \quad \cfrac{\cfrac{[\neg \exists x Px] \quad [\exists x Px]}{\forall x Px}}{\exists x Px \to \forall x Px}}{\exists x Px}}{\exists x \neg Px \to \exists x Px}$$

(iii) $\exists x Rxx, \forall x \forall y (Rxy \wedge Ryy \to x = y) \vdash \exists x \forall y (Ryy \to x = y)$

$$\dfrac{\exists x Rxx \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Raa}{Raa \wedge Raa} \qquad \dfrac{\dfrac{\forall x \forall y (Rxy \wedge Ryy \to x = y)}{\forall y (Ray \wedge Ryy \to a = y)}}{Raa \wedge Raa \to a = a}}{a = a}}{Raa \to a = a}}{\forall y (Ryy \to a = y)}}{\exists x \forall y\, (Ryy \to x = y)}}{\exists x \forall y\, (Ryy \to x = y)}$$

Answer. This claim is false. Here is a counterexample:

$$D_{\mathscr{S}} = \{1, 2\}$$
$$|R|_{\mathscr{S}} = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$$

5. (a) Use the following dictionary to translate each of the following sentences of $\mathscr{L}_2$ into idiomatic English: [7]

P: … is a critic
Q: … is knowledgeable
R: … admires …

(1) $\forall x\, (\exists y\, (Rxy \wedge Py) \to Px)$

Only critics admire critics.

(2) $\forall x\, (Px \to Qx)$

Every critic is knowledgeable.

(3) $\exists x\, (Qx \wedge \forall y \neg (Rxy \wedge Py))$

Someone knowledgeable admires no critics.

(4) $\exists x\, Rxx$

Someone admires oneself.

(5) $\exists x \neg Rxx$

Someone does not admire oneself.

[18] (b) Find any $\mathscr{L}_2$-structures satisfying each of the following conditions. There is no need to prove that your structure satisfies the relevant condition; you need only specify the structure and briefly sketch the reasons why your proposed structure satisfies the condition. If there is no structure satisfying the relevant condition, then justify your answer.

(i) Each of (1), (3) and (5) in part (a) are true and (2) and (4) are false.

$$\begin{aligned} D_{\mathscr{S}} &= \{1,2\} \\ |P|_{\mathscr{S}} &= \{1\} \\ |Q|_{\mathscr{S}} &= \{2\} \\ |R|_{\mathscr{S}} &= \{\} \end{aligned}$$

(ii) Each of (1), (2) and (4) in part (a) are true and (3) and (5) are false.

$$\begin{aligned} D_{\mathscr{S}} &= \{1,2\} \\ |P|_{\mathscr{S}} &= \{1\} \\ |Q|_{\mathscr{S}} &= \{1\} \\ |R|_{\mathscr{S}} &= \{\langle 1,1\rangle, \langle 2,2\rangle\} \end{aligned}$$

(iii) Each of (2), (3) and (4) in part (a) are true and (1) and (5) are false.

$$\begin{aligned} D_{\mathscr{S}} &= \{1,2\} \\ |P|_{\mathscr{S}} &= \{1\} \\ |Q|_{\mathscr{S}} &= \{1,2\} \\ |R|_{\mathscr{S}} &= \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle\} \end{aligned}$$

(iv) Each of (2), (4) and (5) in part (a) are true and (1) and (3) are false.

$$\begin{aligned} D_{\mathscr{S}} &= \{1,2\} \\ |P|_{\mathscr{S}} &= \{2\} \\ |Q|_{\mathscr{S}} &= \{1,2\} \\ |R|_{\mathscr{S}} &= \{\langle 1,2\rangle, \langle 2,2\rangle\} \end{aligned}$$

(v) Every $\mathscr{L}_2$-sentence in part (a) is true.

$$\begin{aligned} D_{\mathscr{S}} &= \{1,2\} \\ |P|_{\mathscr{S}} &= \{1\} \\ |Q|_{\mathscr{S}} &= \{1,2\} \\ |R|_{\mathscr{S}} &= \{\langle 1,1\rangle\} \end{aligned}$$

(vi) Every $\mathscr{L}_2$-sentence in part (a) is false. *There is no such structure as (4) is entailed by the negation of (5).*

$$\dfrac{\neg\exists x\neg Rxx \qquad \dfrac{\neg Raa}{\exists xRxx}}{\dfrac{Raa}{\exists xRxx}}$$

6. (a) Add quotation marks to the following sentences in order to obtain true non-ambiguous English sentences whenever possible. Comment on any points of interest and specify whether you think there is more than one way—or none—to answer the question. [8]

(i) α, β, and γ are Greek letters.

‘α’, ‘β’, and ‘γ’ are Greek letters.

(ii) Paris is identical to London is a sentence.

‘Paris is identical to London’ is a sentence

(iii) and are quotation marks.

“’ and ‘” are quotation marks.

(iv) This is the first word of this incomplete sentence.

‘This’ is the first word of ‘this incomplete sentence’

(b) Define the notion of a sentence of the language $\mathscr{L}_1$ of propositional logic. [5]

(c) The negation of a $\mathscr{L}_1$-sentence is the sentence preceded by the symbol ‘$\neg$’. Determine for each of the following relations whether it is reflexive on the set of $\mathscr{L}_1$-sentences, whether it is symmetric on this set and whether it is transitive on it. [6]

(i) the set of all ordered pairs $\langle e, f\rangle$ where f is the negation of e

(ii) the set of all ordered pairs $\langle e, f\rangle$ where f is the negation of e or e is the negation of f

(iii) the set of all ordered pairs $\langle e, f \rangle$ where the set $\{e, f\}$ is semantically consistent.

[6] (d) For each of the sentences below, explain the ways in which it is ambiguous. Whenever possible, reveal the ambiguity by formalising each sentence in two (or more) different ways in $\mathscr{L}_2$. Specify your dictionary carefully and note any points of interest.

(i) All that glitters is not gold.

Answer.

$\forall x(Px \to \neg Qx)$

$\neg\forall x(Px \to Qx)$

(ii) There is a solution to every problem.

Answer.

$\forall x(Px \to \exists y Rxy)$

$\exists y \forall x(Px \to Rxy)$

(iii) No one is allowed to have ice cream and cake.

Answer.

$\forall x(P_1 x \to \neg(\exists y(Py \land Rxy) \land \exists z(Qz \land Rxz)))$

$\forall x(P_1 x \to (\neg\exists y(Py \land Rxy) \land \neg\exists z(Qz \land Rxz)))$

[8] 7. (a) Formalise the following sentences in $\mathscr{L}_=$ using the following dictionary:

P: … is an English county
Q: … is in the South West
R: … is adjacent to …
a: Oxfordshire
b: Berkshire
c: Cumbria

(i) Cumbria is an English county which is not adjacent to Oxfordshire or Berkshire.

$Pc \land \neg Rca \land \neg Rcb$

(ii) No English county is adjacent both to Cumbria and to Oxfordshire, though some English counties are adjacent both to Oxfordshire and Berkshire.

$\forall x(Px \to \neg(Rxc \land Rxa)) \land \exists x(Px \land (Rxa \land Rxb))$

(iii) Every English county has at least two adjacent counties.

$\forall x(Px \to \exists y \exists z((Py \land Pz) \land (\neg x = y \land \neg x = z) \land Rxy \land Rxz)$

(iv) One English county in the South West has only one adjacent county.

$\exists x(Px \land Qx \land \exists y(Py \land Rxy \land \forall z(Pz \land Rxz \to y = z)))$

(b) Consider an $\mathscr{L}_=$-structure $\mathscr{S}$ satisfying the following conditions:

$$D_{\mathscr{S}} = \{x : x \text{ is an English county}\}$$
$$|P|_{\mathscr{S}} = \{x : x \text{ is in the South East}\}$$
$$|R|_{\mathscr{S}} = \{\langle e, d\rangle : e \text{ is adjacent to } d\}$$

Let α be a variable assignment with:

$$|x|^{\alpha}_{\mathscr{S}} = \text{Oxfordshire}$$
$$|y|^{\alpha}_{\mathscr{S}} = \text{Berkshire}$$
$$|z|^{\alpha}_{\mathscr{S}} = \text{Cumbria}$$

Does α satisfy any of the following formulae in $\mathscr{S}$? Justify your answer. (You do not have to give a complete proof, but you should say why you think each formula is or not satisfied in $\mathscr{S}$ by the assignment.) [9]

(i) $\exists x \forall y Rxy$

Answer. The sentence is not satisfied by α—or any other assignment—as no English county is adjacent to every other county.

(ii) $\forall x \exists y Rxy \to Px$

Answer. The conditional is satisfied in the structure by the assignment α because Oxfordshire is in the South East.

(iii) $\forall x(\exists y Rxy \to Px)$

Answer. The open formula $\exists y Rxy \to Px$ is not satisfied in the structure by the assignment β which differs from α only with respect to x, where $|x|^{\beta}_{\mathscr{S}}$ = Oxfordshire. Therefore the sentence is not satisfied by the original assignment α.

(c) For each of the sentences below, explain the ways in which it is ambiguous. Whenever possible, reveal the ambiguity by formalising each sentence in two (or more) different ways in $\mathscr{L}_=$. Make sure to specify your dictionary carefully and note any points of interest. [8]

(i) No two people own the same car.

Answer. *This sentence has at least two readings depending on whether 'the same' is read as expressing qualitative or numerical identity.*

P: ... is a person
Q: ... is a car
R: ... owns ...
R_1: ... is qualitatively similar to ...

$\forall x \forall y \forall z_1 \forall z_2 ((Px \wedge Qz_1 \wedge Rxz_1) \wedge (Py \wedge Qz_2 \wedge Ryz_2) \rightarrow z_1 = z_2)$

$\forall x \forall y \forall z_1 \forall z_2 ((Px \wedge Qz_1 \wedge Rxz_1) \wedge (Py \wedge Qz_2 \wedge Ryz_2) \rightarrow R_1 z_1, z_2)$

(ii) The man on the moon is not an astronaut.

Answer. *This sentence has at least two readings depending on whether negation takes narrow or wide scope*:

P: ... is a man
Q: ... is on the moon
R: ... is an astronaut

$\exists x (Px \wedge Qx \wedge \forall y ((Py \wedge Qy) \rightarrow x = y) \wedge \neg Rx)$

$\neg \exists x (Px \wedge Qx \wedge \forall y ((Py \wedge Qy) \rightarrow x = y) \wedge Rx)$