Collection paper

INTRODUCTION TO LOGIC

Hilary Term 2009

TIME:

Please answer questions.

I have indicated in the margin how many points I would give for each question. The maximum for each question is 25.

1. (a) What does it mean for an argument in English to be propositionally valid? 2

   Answer. An argument in English is propositionally valid if and only if its formalisation in $\mathcal{L}_1$ is valid.

   (b) If the conclusion of an English argument is a tautology, can the argument be valid? And can it be not valid? 2

   Answer. In this case the argument will always be valid because the conclusion will be true under any interpretation.

   (c) What is the scope of an occurrence of connective in a sentence of the language $\mathcal{L}_1$ of propositional logic? 2

   Answer. The scope of an occurrence of a connective in a sentence ϕ is (the occurrence of) the smallest subsentence of ϕ that contains this occurrence of the connective.

   (d) Determine the scopes of the underlined occurrences of quantifiers after adding any brackets that have been omitted in accordance with the rules for saving brackets. 3

   (i) $P \to Q \vee R_{23} \underline{\vee} R_{23}$

   (ii) $\neg\underline{\neg}(P \underline{\leftrightarrow} Q \wedge P_3) \vee (P_2 \wedge \neg R)$

   Answer.

   (i) $(P \to \underbrace{((Q \vee R_{23}) \underline{\vee} R_{23})}_{scope})$

   [Explanation: the bracketing conventions force left-bracketing of the three disjuncts.]

(ii) $(\neg \underbrace{\underline{\neg}\,\overbrace{(P \underline{\leftrightarrow} (Q \wedge P_3))}^{\textit{scope of} \leftrightarrow}}_{\textit{scope of} \neg} \vee (P_2 \wedge \neg R))$

(e) Show that the following argument can be transformed into a propositionally valid argument if the premisses are appropriately reformulated and if premisses are added on which one may naturally rely. You may use the truth table method or give a proof in Natural Deduction. Specify your dictionary carefully and note any difficulties or points of interest.

formalisation 8
truth table: 4
comments:4

> If Brown sold his old Ford, he must have used the money for a trip to the Caribbean and he must be there now. Otherwise he could only have afforded a trip to Barcelona and he must be there by now. Nobody else would buy the beaten-up banger from Brown; only Jones could have bought that old Ford. So either Jones bought Brown's Ford or Brown is in Barcelona but it can't be the case that both are true, that is, that Jones bought the Ford and Brown is in Barcelona.

Answer.

P:	*Brown sold his old Ford*
Q_1:	*Brown is in the Caribbean*
Q_2:	*Brown used the money for a trip to the Caribbean*
R_1:	*Brown is in Barcelona*
R_2:	*Brown can afford a trip to Barcelona*
R:	*Jones bought Brown's old Ford*

Formalisation:
First premiss: $P \to Q_2 \wedge Q_1$
Second premiss: $\neg P \to R_2 \wedge R_1$
Third premiss: I reformulate the premiss'Nobody else would buy the beaten-up banger from Brown; only Jones could have bought that old Ford' as 'if Brown sold his old Ford then Jones bought Browns old Ford and the reason for this is that nobody else would have bought Brown's old Ford'. Moreover, I add the extra premiss that if Jones bought then Brown sold his old Ford. Combining this premiss with stated premiss I formalise the third premiss together with the additional premiss as $P \leftrightarrow R$. *Of course the additional premiss can be formalised separately.*
Additional premiss: To get the second part of the conclusion, an additional premiss is needed. It isn't possible that Brown is in the Caribbean and *in Barcelona. So I add the additional premiss that it is not the case that Brown is in the Caribbean and that Brown is in Barcelona. This is formalised as* $\neg(Q_1 \wedge R_1)$ *Conclusion:*
$(R \vee R_1) \wedge \neg(R \wedge R_1)$

In these formalisations I have formalised 'if …, then' as the arrow, which is at least controversial.

No one has to show that the resulting formal argument is valid:

$$P \to Q_2 \wedge Q_1, \neg P \to R_2 \wedge R_1, P \leftrightarrow R, \neg(Q_1 \wedge R_1) \vdash (R \vee R_1) \wedge \neg(R \wedge R_1)$$

This claim can be established using a partial truth table or a proof in Natural Deduction:

$$\dfrac{[\neg(R \vee R_1)] \qquad \dfrac{\dfrac{[\neg(R \vee R_1)] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad \dfrac{P \leftrightarrow R}{P \to R}}{R} \qquad [\neg R]}{\neg P} \qquad \neg P \to R_2 \wedge R_1}{R_2 \wedge R_1}}{R_1}}{R \vee R_1}}{R}}{R \vee R_1}}{R \vee R_1}$$

$$\dfrac{\dfrac{[R \wedge R_1]}{R_1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[R \wedge R_1]}{R} \qquad \dfrac{P \leftrightarrow R}{R \to P}}{P} \qquad P \to Q_2 \wedge Q_1}{Q_2 \wedge Q_2}}{Q_1} \qquad [R_1]}{Q_1 \wedge R_1} \qquad \neg(Q_1 \wedge R_1)}{\neg R_1}}{\neg(R \wedge R_1)}$$

Finally the two proofs are merged into one using the rule for introducing $\wedge$*. This yields the conclusion* $(R \vee R_1) \wedge \neg(R \wedge R_1)$*.*

2. (a) Establish each of the following claims by means of proofs in the system of Natural Deduction:

(i) $P \to (Q_1 \lor Q_2), \neg Q_1 \land \neg Q_2 \vdash \neg P$ 5

Answer. Proof:

$$\dfrac{\dfrac{\dfrac{P \to (Q_1 \lor Q_2) \qquad [P]}{Q_1 \lor Q_2} \qquad \dfrac{[Q_1] \qquad \dfrac{[\neg Q_1 \land \neg Q_2]}{\neg Q_1}}{\neg(\neg Q_1 \land \neg Q_2)} \qquad \dfrac{[Q_2] \qquad \dfrac{[\neg Q_1 \land \neg Q_2]}{\neg Q_2}}{\neg(\neg Q_1 \land \neg Q_2)}}{\neg(\neg Q_1 \land \neg Q_2)} \qquad \neg Q_1 \land \neg Q_2}{\neg P}$$

(ii) $\forall x\, Qx \lor \forall x\, Rx \vdash \forall x\, (Qx \lor Rx)$ 5

Answer. Proof:

$$\dfrac{\forall x\, Qx \lor \forall x\, Rx \qquad \dfrac{\dfrac{\dfrac{[\forall x\, Qx]}{Qa}}{Qa \lor Ra}}{\forall x\,(Qx \lor Rx)} \qquad \dfrac{\dfrac{\dfrac{[\forall x\, Rx]}{Ra}}{Qa \lor Ra}}{\forall x\,(Qx \lor Rx)}}{\forall x\,(Qx \lor Rx)}$$

(iii) $\exists x\, (Px \to \forall y\, Ryy) \vdash \forall x\, Px \to Raa$ 5

Answer. Proof:

$$\dfrac{\exists x (Px \to \forall y Ryy) \qquad \dfrac{\dfrac{\dfrac{[Pb \to \forall y Ryy] \qquad \dfrac{[\forall x Px]}{Pb}}{\forall y Ryy}}{Raa}}{\forall x Px \to Raa}}{\forall x Px \to Raa}$$

(b) Explain why the following attempted proofs are not correct proofs in the system of Natural Deduction. Note all steps that are not correct. Give complete correct proofs for any true claims below and counterexamples to any false claims you find. 2

(i) $P \lor Q, P \leftrightarrow Q \vdash P \land Q$

$$\dfrac{P \lor Q \qquad \dfrac{[P] \qquad [Q]}{P \land Q} \qquad \dfrac{[P] \qquad [Q]}{P \land Q}}{P \land Q}$$

Answer. The rule $\lor$*Elim doesn't allow one to discharge* P *and* Q *on a branch. One may only discharge* P *on one branch and* Q *on the other branch.*

Correct proof:

$$\dfrac{P \lor Q \qquad \dfrac{[P] \qquad \dfrac{[P] \qquad \dfrac{P \leftrightarrow Q}{P \to Q}}{Q}}{P \land Q} \qquad \dfrac{\dfrac{[Q] \qquad \dfrac{P \leftrightarrow Q}{Q \to P}}{P} \qquad [Q]}{P \land Q}}{P \land Q}$$

(ii) $\forall x\,(Px \lor Qx) \vdash \forall x\,Px \lor \forall x\,Qx$ 4

$$\dfrac{\dfrac{\forall x\,(Px \lor Qx)}{Pa \lor Qa} \qquad \dfrac{\dfrac{[Pa]}{\forall x\,Px}}{\forall x\,Px \lor \forall x\,Qx} \qquad \dfrac{\dfrac{[Qa]}{\forall x\,Qx}}{\forall x\,Px \lor \forall x\,Qx}}{\forall x\,Px \lor \forall x\,Qx}$$

Answer. *Both applications of $\forall$Intro are not correct. The rule $\forall$Intro states:*

> *Assume that ϕ is a formula with at most v occurring freely and that ϕ does not contain the constant t. Assume further that there is a proof of $\phi[t/v]$ in which t does not occur in any undischarged assumption. Then the result of appending $\forall v\,\phi$ to that proof is a proof of $\forall v\,\phi$.*

The constant t is the constant a in the present case, which does occur in the undischarged assumptions Pa and Qa, which is not permitted by the rule.

This claim is false. Here is a counterexample:

$$D_{\mathcal{S}} = \{1, 2\}$$
$$|P|_{\mathcal{S}} = \{1\}$$
$$|Q|_{\mathcal{S}} = \{2\}$$

(iii) $\forall z_1\,\forall z_2\,(Rz_1z_2 \to Qz_2z_1), \exists x\,\exists y\,Rxy \vdash \exists z\,\exists y\,Qzy$ 4

$$\dfrac{\exists x\,\exists y\,Rxy \qquad \dfrac{\dfrac{[\exists y\,Rby] \qquad \dfrac{\dfrac{[Rba] \qquad \dfrac{\forall z_1\,\forall z_2\,(Rz_1z_2 \to Qz_2z_1)}{Rba \to Qab}}{Qab}}{\exists y\,Qay}}{\exists y\,Qay}}{\exists z\,\exists y\,Qzy}}{\exists z\,\exists y\,Qzy}$$

Answer. *The step from $\forall z_1\,\forall z_2\,(Rz_1z_2 \to Qz_2z_1)$ to $Rba \to Qab$ isn't covered by any rule (or it's applying $\forall$Elim 'twice' in one step.*

The other problem is that in the step from $\exists y\,Rby$ and $\exists y\,Qay$ to $\exists y\,Qay$ the rule $\exists$Elim isn't correctly applied. The rule reads as follows:

> *Assume that ϕ is a formula with at most v occurring freely and that the constant t does not occur in ϕ. Assume further that there is a proof of the sentence ψ in which t does not occur in any undischarged assumption other than $\phi[t/v]$. Then the result of appending ψ to a proof of $\exists v\,\phi$ and the proof of ψ and of discharging all assumptions of $\phi[t/v]$ in the proof of ψ is a proof of ψ.*

The rule doesn't allow the constant t (here a) to occur in ψ (here $\exists y\,Qay$). Only once a has disappeared (through an application of ∃Intro) one can apply ∃Elim.

Correct proof:

$$\dfrac{\exists x\,\exists y\,Rxy \qquad \dfrac{\exists y\,Rby \qquad \dfrac{\dfrac{\dfrac{[Rba] \qquad \dfrac{\dfrac{\forall z_1\,\forall z_2\,(Rz_1z_2 \to Qz_2z_1)}{\forall z_2\,(Rbz_2 \to Qz_2b)}}{Rba \to Qab}}{Qab}}{\exists y\,Qay}}{\exists z\,\exists y\,Qzy}}{\exists z\,\exists y\,Qzy}}{\exists z\,\exists y\,Qzy}$$

3. (a) Add quotation marks to the following expressions so that true and non-ambiguous English sentences are obtained if possible. Comment on any difficulties and indicate if there is more than one way to answer the question.

(i) The quotation of the quotation of ! is !. 2

Answer. The quotation of the quotation of ‘!’ is “‘!’”.

The quotation of the quotation of “!” is “‘“!”’”.

And so on.

(ii) $(P \to \neg Q)$ is a sentence of the language of propositional logic. 2

Answer. ‘$(P \to \neg Q)$’ *is a sentence of the language of propositional logic.*

(iii) is an opening quotation mark. 2

Answer. ‘“’is an opening quotation mark.

(iv) It’s raining and it’s snowing and it’s cold are English sentences. 2

Answer. ‘It’s raining and it’s snowing’ and ‘it’s cold’ are English sentences.

‘It’s raining’ and ‘it’s snowing and it’s cold’ are English sentences.

‘It’s raining’ and ‘it’s snowing’ and ‘it’s cold’ are English sentences. (One might object that the first two sentences in quotation marks should be connected by a comma not by ‘and’.)

The following is not a possible solution:

> *‘It’s raining and it’s snowing and it’s cold’ are English sentences.*

because the plural ‘are’ wouldn’t be correct.

If somebody argued that, e.g., ‘it’s snowing’ isn’t an English sentence because the full stop is missing and/or the first word isn’t capitalised, I would accept this. Then there is no solution.

The example shows that American (double) quotation marks are not so bad after all as they cannot be confused with the apostrophe.

(b) Consider the relation Q having all ordered pairs $\langle d, e \rangle$ as elements where d is the quotation of e. Answer the following questions and substantiate your answers:

(i) Is Q reflexive on the set of all strings of English expressions? 2

Answer. No: ‘This is an expression.’ is not the quotation of ‘This is an expression.’

(ii) Is Q transitive? 2

Answer. No: '"This is an expression."' is the quotation of "This is an expression.", which is the quotation of 'This is an expression.', but the first expression is not the quotation of the last expression. 3

(iii) Is Q symmetric? Is Q antisymmetric? Is Q asymmetric?

Answer. It's not symmetric, but antisymmetric and asymmetric. The quotation of an expression has always two more symbols that the expression itself (namely an opening and a closing quotation mark). So it cannot be symmetric, but it is asymmetric and therefore also antisymmetric.

(c) The language $\mathcal{L}_A$ is defined as follows (in the definition I have dropped quotation marks in accordance with the usual convention):

The letters A and B are sentences of $\mathcal{L}_A$.
If ϕ and ψ are sentences of $\mathcal{L}_A$, then $N\phi$ and $(I\phi\psi)$ are sentences of $\mathcal{L}_A$.
Nothing else is a sentence of $\mathcal{L}_A$.

(i) What are the Greek letters ϕ and ψ called if used as above? What is their use? 2

Answer. Here 'ϕ' and 'ψ' are metavariables. They are not expression of $\mathcal{L}_A$ but range over expressions of $\mathcal{L}_A$. So here they are used for making the general claim, if an expression is a sentence of $\mathcal{L}_A$ then the result of attaching the symbol N in front of it yields a sentence of $\mathcal{L}_A$ (and similarly for I). 2

(ii) Is $(I(IAA)A)$ a sentence of $\mathcal{L}_A$? Substantiate your answer.

Answer. A is a sentence of $\mathcal{L}_A$. Hence IAA is a sentence of $\mathcal{L}_A$. This yields the claim. 2

(iii) Is the expression $N(INNANNB)$ a sentence of $\mathcal{L}_A$? Substantiate your answer.

Answer. A is a a sentence of $\mathcal{L}_A$, so NA is a sentence of $\mathcal{L}_A$ and thus NNA is a sentence of $\mathcal{L}_A$. Similarly, NNB is a sentence of $\mathcal{L}_A$. Therefore $(INNANNB)$ is a sentence of $\mathcal{L}_A$ and also $N(INNANNB)$.

(iv) State rules for saving brackets in such a way that every abbreviation of an 4
$\mathcal{L}_A$-sentence abbreviates at most one $\mathcal{L}_A$-sentence. Try to state a rule or rules that allow one to save as many brackets as possible. Explain why abbreviations do not abbreviate more than one sentence.

Answer. All brackets may be dropped. In an $\mathcal{L}_A$-sentence there must always be a left bracket in front of any occurrence of the symbol I.

The key observation is that the bracketing in a sentence like $IAINAB$ is unique, namely $(IA(INAB))$ whereas it isn't if the brackets are dropped in the usual infix notation; that is, e.g., $P \wedge Q \vee R$ can be bracketed as $((P \wedge Q) \vee R)$ or as $(P \wedge (Q \vee R))$ if no further conventions are applied. Therefore brackets are not needed in $\mathcal{L}_A$-sentences

Remark: In a rigorous proof one would show that if an expression abbreviates sentence ϕ and ψ, then ϕ and ψ must be identical. The proof resembles the usual proof of unique readability by induction on the build-up of ϕ and ψ. The main point is that if a sentence ϕ is an initial segment of a sentence ψ then ϕ and ψ must have the same length and in fact be the same sentence.

4. (a) Formalise the following sentences in the language $\mathcal{L}_=$ of predicate logic with identity using the following dictionary:

P: … is a book
Q: … is on Bill's desk
R: … is red

(i) The book on Bill's desk is red. 2

Answer. $\exists x\,(Px \land Qx \land \forall y\,(Py \land Qy \to y{=}x) \land Rx)$

(ii) Something on Bill's desk isn't red. 2

Answer. $\exists x\,(Qx \land \neg Rx)$

(iii) There is something on Bill's desk that isn't a book. 2

Answer. $\exists x\,(Qx \land \neg Px)$

(iv) There are at least three books on Bill's desk. 2

Answer. $\exists x\,\exists y\,\exists z\,(Px \land Py \land Pz \land Qx \land Qy \land Qz \land \neg x{=}y \land \neg y{=}z \land \neg x{=}z)$

(v) There are at most two red things. 2

Answer. $\forall x\,\forall y\,\forall z\,(Rx \land Ry \land Rz \to x{=}y \lor y{=}z \lor x{=}z)$

(b) Show that the English argument with (i) and (ii) as premises and (iii) as conclusion is valid in predicate logic with identity. 10

Answer. I need to show that the formalisation of the argument in $\mathcal{L}_=$ is valid, that is, I need to show that

$$\exists x\,(Px \land Qx \land \forall y\,(Py \land Qy \to y{=}x) \land Rx), \exists x\,(Qx \land \neg Rx) \vdash \exists x\,(Qx \land \neg Px)$$

This claim can be established by the proof on page. 12.

(c) Show that the formalisation of the argument with (i) and (iii) as premisses and (ii) as conclusion is not valid in predicate logic with identity by providing a counterexample. 5

Answer. I show that the formalisation of this argument in $\mathcal{L}_=$ isn't valid, that is, I show

$$\exists x\,(Px \land Qx \land \forall y\,(Py \land Qy \to y{=}x) \land Rx), \exists x\,(Qx \land \neg Px) \nvDash \exists x\,(Qx \land \neg Rx)$$

The following is a counterexample:

$$D_{\mathcal{S}} = \{1, 2\}$$
$$|P|_{\mathcal{S}} = \{1\}$$
$$|Q|_{\mathcal{S}} = \{1, 2\}$$
$$|R|_{\mathcal{S}} = \{1, 2\}$$

5. (a) Show that the following argument is propositionally valid if it is suitably formalised. Note any points of interest. formalisation 9 truth table: 4

   If ϕ implies ψ and ψ implies χ then ϕ and χ share some sentence letter, unless ϕ is a contradiction or χ is logically true. Therefore, ϕ is a contradiction or ϕ doesn't imply ψ or ψ doesn't imply χ, if ϕ and χ don't share a sentence letter, provided that χ is not logically true.

   Answer.

   I use the following dictionary:

P:	*ϕ implies ψ*
Q:	*ψ implies χ*
R:	*ϕ and χ share some sentence letter*
P_1:	*ϕ is a contradiction*
Q_1:	*χ is logically true*

   Formalisation:

   $$(P \land Q \to R) \lor (P_1 \lor Q_1) \vdash \neg Q_1 \to (\neg R \to P_1 \lor \neg P \lor \neg Q)$$

   This can be proved by a partial truth table. I skip it as they are awkward to typeset.

   (b) Determine for each of the following relations
   - whether it is reflexive on the set of all $\mathcal{L}_2$-sentences,
   - whether it is symmetric,
   - whether it is antisymmetric,
   - whether it is asymmetric, and
   - whether it is transitive.

   Substantiate your answers. In the following ϕ and ψ are understood to be $\mathcal{L}_1$-sentences.

   (i) The set of all pairs $\langle \phi, \psi \rangle$ such that $\phi \to \psi$ is a contradiction. 4

   Answer. This relation is not reflexive on the set of all $\mathcal{L}_2$-sentences ($P \to P$ is not a contradiction).

$$
\dfrac{\exists x \, (Px \land Qx \land \forall y \, (Py \land Qy \to y = x) \land Rx) \qquad \dfrac{\exists x \, (Qx \land \neg Rx) \qquad \dfrac{\dfrac{\dfrac{[Qb \land \neg Rb]}{Qb} \qquad \dfrac{\dfrac{\dfrac{[Pa \land Qa \land \forall y \, (Py \land Qy \to y = a) \land Ra]}{Ra} \qquad \dfrac{\dfrac{[Pb] \qquad \dfrac{[Qb \land \neg Rb]}{Qb}}{Pb \land Qb} \qquad \dfrac{\dfrac{\dfrac{[Pa \land Qa \land \forall y \, (Py \land Qy \to y = a) \land Ra]}{Pa \land Qa \land \forall y \, (Py \land Qy \to y = a)}}{\forall y \, (Py \land Qy \to y = a)}}{Pb \land Qb \to b = a}}{b = a}}{Rb} \qquad \dfrac{[Qb \land \neg Rb]}{\neg Rb}}{\neg Pb}}{Qb \land \neg Pb}}{\exists x \, (Qx \land \neg Px)}}{\exists x \, (Qx \land \neg Px)}}{\exists x \, (Qx \land \neg Px)}
$$

It's not symmetric ($P \lor \neg P \to P \land \neg P$ is a contradiction, $P \land \neg P \to P \lor \neg P$ is not).

It's asymmetric and thus also antisymmetric because if $\phi \to \psi$ is a contradiction then in all $\mathcal{L}_2$-structures ϕ is true and ψ is false, so $\psi \to \phi$ can't be a contradiction.

The relation is transitive because it's not possible that $\phi \to \psi$ and $\psi \to \chi$ are both contradictions as ψ would have to be true and false in all structures.

(ii) The set of all pairs $\langle \phi, \psi \rangle$ such that $\phi \to \psi$ is a logical truth. 4

Answer. The relation is reflexive on the set of all $\mathcal{L}_2$-sentences because $\phi \to \phi$ is true in all $\mathcal{L}_2$-structures for all sentences ϕ of $\mathcal{L}_2$.

It's not symmetric because $P \land \neg P \to P$ is a logical true while $P \to P \land \neg P$ is not.

It's neither asymmetric nor antisymmetric $P \land Q \to Q \land P$ is logically true as is $Q \land P \to P \land Q$.

The relation is transitive. Assume $\phi \to \psi$ and $\phi \to \chi$ are both logically true Then χ is true in any structure in which ψ is true, which in turn in true in any structure in which ϕ i true. Thus χ is true in any structure in which ϕ is true and $\phi \to \chi$ is logically true.

(iii) The set of all pairs $\langle \phi, \psi \rangle$ such that for some $\mathcal{L}_2$-logical truth χ, $\psi \land \phi \vDash \chi$. 2

Answer. This is the set of all pairs $\langle \phi, \psi \rangle$ where ϕ and ψ are $\mathcal{L}_2$-sentences. This relation is reflexive, symmetric (thus not asymmetric or antisymmetric), and transitive.

(iv) The set of all pairs $\langle \phi, \psi \rangle$ such that for every $\mathcal{L}_2$-logical truth χ, $\psi \land \phi \vDash \chi$. 2

Answer. This is again the set of all pairs $\langle \phi, \psi \rangle$ where ϕ and ψ are $\mathcal{L}_2$-sentences. This relation is reflexive, symmetric (thus not asymmetric or antisymmetric), and transitive.

6. (a) How is an $\mathcal{L}_2$-structure defined? 2

Answer. An $\mathcal{L}_2$-structure is an ordered pair $\langle D, I\rangle$ where D is some non-empty set and I is a function from the set of all constants, sentence letters and predicate letters such that the value of every constant is an element of D, the value of every sentence letter is a truth-value T *or* F, *and the value of every n-ary predicate letter is an n-ary relation.*

(b) What are the semantic values (extensions) of constants in an $\mathcal{L}_2$-structure? 1

Answer. Objects from the domain of the structure.

(c) What is a variable assignment over an $\mathcal{L}_2$-structure? 2

Answer. A variable assignment over an $\mathcal{L}_2$-structure $\mathcal{A}$ is a function assigning to each variable an element of the domain $D_{\mathcal{A}}$ of $\mathcal{A}$.

(d) Consider the following $\mathcal{L}_2$-structure $\mathcal{S}$:

$$\begin{aligned} D_{\mathcal{S}} &= \{d : d \text{ is a planet of the solar system}\} \\ |P|_{\mathcal{S}} &= \{\text{Mercury, Venus}\} \\ |Q|_{\mathcal{S}} &= \{\langle e, d\rangle : e \text{ is larger than } d\} \\ |R|_{\mathcal{S}} &= \{\langle e, d\rangle : e \text{ is farther from the sun than } d\} \end{aligned}$$

Sorry for the typo: it should read 'e is larger than d' and 'e is farther from the sun than d'.

Let α be a variable assignment assigning Venus to x and Jupiter to y.

(1) Which of the following formulas are satisfied by α in $\mathcal{S}$? Explain your answers. 3

(i) $Qxy \wedge Ryx$

Answer. Venus isn't larger than Jupiter so $\langle$Venus, Jupiter$\rangle \notin |Q|_{\mathcal{S}}$. Therefore $\langle |x|^{\alpha}_{\mathcal{S}}, |y|^{\alpha}_{\mathcal{S}}\rangle \notin |Q|_{\mathcal{S}}$ and hence $|Qxy|^{\alpha}_{\mathcal{S}} =$ F and consequently also $|Qxy \wedge Ryx|^{\alpha}_{\mathcal{S}} =$ F. So this formula is not satisfied by α in $\mathcal{S}$.

(ii) $Rxy \rightarrow Qyx$ 3

Answer. Venus isn't larger than Jupiter so $\langle$Venus, Jupiter$\rangle \notin |Q|_{\mathcal{S}}$. Therefore $\langle |x|^{\alpha}_{\mathcal{S}}, |y|^{\alpha}_{\mathcal{S}}\rangle \notin |Q|_{\mathcal{S}}$ and hence $|Qxy|^{\alpha}_{\mathcal{S}} =$ F and consequently $|Qxy \rightarrow Ryx|^{\alpha}_{\mathcal{S}} =$ T. So this formula is satisfied by α in $\mathcal{S}$.

(iii) $\exists y\,(Py \leftrightarrow Ryx) \rightarrow Py$ 3

Answer. I show that $|\exists y\,(Py \leftrightarrow Ryx)|^{\alpha}_{\mathcal{S}} =$ F. If otherwise there must be a variable assignment differing from α in y at most such that $|Py \leftrightarrow Ryx|^{\beta}_{\mathcal{S}} =$ T. But, on the one hand, if $|Py|^{\beta}_{\mathcal{S}} =$ T, then $|y|^{\beta}_{\mathcal{S}}$ bust be Mercury or Venus, which are both not farther from the sun that Venus, so $|Ryx|^{\beta}_{\mathcal{S}} =$ F and thus $|Py \leftrightarrow Ryx|^{\beta}_{\mathcal{S}} =$ F. But if, on the other hand,

$|Py|^\beta_\mathcal{S}$ = F *then* $|y|^\beta_\mathcal{S}$ *must be a planet other than Mercury or Venus, that is, a planet that it farther away from the sun than Venus and thus* $|Ryx|^\beta_\mathcal{S}$ = T, *and so* $|Py \leftrightarrow Ryx|^\beta_\mathcal{S}$ = F. *Hence there is no variable assignment* β *differing from* α *in* y *at most such that* $|Py \leftrightarrow Ryx|^\beta_\mathcal{S}$ = T. *So* $|\exists y\,(Py \leftrightarrow Ryx|^\alpha_\mathcal{S}$ = F) *and, therefore,* $|\exists y\,(Py \leftrightarrow Ryx|^\alpha_\mathcal{S}$ = T.

(2) Which of the following sentences are true in $\mathcal{S}$? Justify your answers as fully as possible.

(i) $\forall x\,\forall y\,(Rxy \to Qyx)$ 3

Answer.

The sentence is false in $\mathcal{S}$.

Informally speaking, the sentence says that the farther away a planet is from the sun, the smaller it is.

Consider Mercury and Jupiter. Jupiter is larger than Mercury but much farther away from the sun. Let α *be a variable assignment over* $\mathcal{S}$ *assigning Jupiter to* x *and Mercury to* y. *Here is now the reasoning in full detail (skipping some steps would be ok):*

$$\langle \textit{Jupiter, Mercury} \rangle \in |R|_\mathcal{S}$$
$$\langle |x|^\alpha_\mathcal{S}, |y|^\alpha_\mathcal{S} \rangle \in |R|_\mathcal{S}$$
$$|Rxy|^\alpha_\mathcal{S} = \text{T}$$
$$\langle \textit{Mercury, Jupiter} \rangle \notin |Q|_\mathcal{S}$$
$$\langle |y|^\alpha_\mathcal{S}, |x|^\alpha_\mathcal{S} \rangle \notin |Q|_\mathcal{S}$$
$$|Qyx|^\alpha_\mathcal{S} = \text{F}$$
$$|Rxy \to Qyx|^\alpha_\mathcal{S} = \text{F}$$
$$|\forall y\,(Rxy \to Qyx)|^\alpha_\mathcal{S} = \text{F}$$
$$|\forall x\,\forall y\,(Rxy \to Qyx)|^\alpha_\mathcal{S} = \text{F}$$

(ii) $\exists y\,\exists x\,(\neg(Py \land Qxy) \land \exists z(Pz \land Qyz))$ 4

Answer. The sentence is true in $\mathcal{S}$.

Let α *be a variable assignment over* $\mathcal{S}$ *that assigns Jupiter to* x, *Uranus to* y *and Venus to* z.

$$
\begin{aligned}
&\textit{Venus} \in |P|_{\mathcal{S}}\\
&|z|^{\alpha}_{\mathcal{S}} \in |P|_{\mathcal{S}}\\
&|Pz|^{\alpha}_{\mathcal{S}} = \mathrm{T}\\
&\langle \textit{Uranus, Venus} \rangle \in |Q|_{\mathcal{S}}\\
&|Qyz|^{\alpha}_{\mathcal{S}} = \mathrm{T}\\
&|Pz \land Qyz|^{\alpha}_{\mathcal{S}} = \mathrm{T}\\
&|\exists z\, Pz \land Qyz|^{\alpha}_{\mathcal{S}} = \mathrm{T}\\
&\textit{Uranus} \notin |P|_{\mathcal{S}}\\
&|y|^{\alpha}_{\mathcal{S}} \notin |P|_{\mathcal{S}}\\
&|Py|^{\alpha}_{\mathcal{S}} = \mathrm{F}\\
&|Py \land Qxy|^{\alpha}_{\mathcal{S}} = \mathrm{F}\\
&|\neg(Py \land Qxy)|^{\alpha}_{\mathcal{S}} = \mathrm{T}\\
&|\neg(Py \land Qxy) \land \exists z\, Pz \land Qyz|^{\alpha}_{\mathcal{S}} = \mathrm{T}\\
&|\exists x\, (\neg(Py \land Qxy) \land \exists z\, Pz \land Qyz)|^{\alpha}_{\mathcal{S}} = \mathrm{T}\\
&|\exists y\, \exists x\, (\neg(Py \land Qxy) \land \exists z\, Pz \land Qyz)|^{\alpha}_{\mathcal{S}} = \mathrm{T}
\end{aligned}
$$

The steps involving the existential quantifier are justified because α differs from α in the respective variable at most (because it doesn't differ at all).

(e) Disprove the following claims by providing counterexamples (no need to prove that your structure is a counterexample; you only need to specify the structure).

(i) $\exists x\, \exists y\, Qxy \vDash \neg\forall x\, \forall y\, Qxy$ 2

Answer. The following structure $\mathcal{A}$ is a counterexample (with the other values fixed arbitrarily):

$$
\begin{aligned}
D_{\mathcal{A}} &= \{1, 2\}\\
|Q|_{\mathcal{A}} &= \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}
\end{aligned}
$$

(ii) $\forall x\, (\neg Px \to \exists y\, (Rxy \lor Ryx)) \vDash \forall x\, \forall y\, (\neg Rxy \land \neg Ryx \to Px)$ 2

Answer. The following structure $\mathcal{B}$ is a counterexample (with the other values fixed arbitrarily):

$$
\begin{aligned}
D_{\mathcal{B}} &= \{1, 2\}\\
|P|_{\mathcal{B}} &= \emptyset\\
|R|_{\mathcal{B}} &= \{\langle 1,2\rangle, \langle 2,1\rangle\}
\end{aligned}
$$

7. *The solutions here are somewhat sketchy as it's difficult to give definitive answers. Most answers would have to be supplemented by appropriate formalisations and dictionaries.*

(a) For each of the sentences below explain the way in which it is ambiguous. If possible, reveal the ambiguity by formalising the sentence in two (or more) different ways.

(i) Some letters can be found in every line. 2

Answer. Scope ambiguity: $\exists y \, \forall x \, Ryx$ *or* $\forall x \, \exists y \, Ryx$*, where* R *is translated as '… is a letter that can be found in line … '.*

There is, perhaps, also an ambiguity between type and token: the sentence may be taken to be about abstract letter and lines or about figures of ink on paper or the like.

(ii) The first line in the book contains 15 letters. 2

Answer. An ambiguity between type and token. Does the first line contain 15 different letters, that is, arbitrarily many printed letters that are of 15 different types, or are the exactly 15 printed symbols in the line?

(iii) In a library in Oxford Albert saw the same book he had seen ten years earlier in a book shop in Budapest. Some edition of this book was known to all of Albert's friends, who haven't been there. 3

Answer. There is an ambiguity between qualitative and numerical identity: does 'same' book really mean there very same copy or just the same title (but possibly a different copy).

Formalisation of the first sentence highlighting this ambiguity would be: $\exists x \, (Px \wedge \forall y \, (Py \to R_1 xy))$ *or* $\exists x \, (Px \wedge \forall y \, (Py \to R_2 xy))$ *where* R_1 *stands for 'is numerically identical to' and* R_2 *for 'is qualitatively identical to', and* P *for 'in a library in Oxford Albert saw … and Albert saw it ten years earlier in Budapest'.*

In the second sentence there is again a structural ambiguity (order of quantifiers).

The reference of there is ambiguous: does it refer to the Oxford library or to Budapest? Can be formalised by different constants.

(b) Formalise the following sentences as detailed as possible in the language $\mathcal{L}_2$ of predicate logic specifying your dictionary. Comment on any difficulties and points or interest.

(i) Tim and Tom painted the wall. 2

Answer. The most natural reading is formalised as Pab *(they painted the wall together, which doesn't imply that Tim (or Tom) painted it (completely). But the sentence may be ambiguous.*

(ii) Tim ran and Tom ran quickly. 2

Answer. $Pa \wedge Qb$ *where P stands for 'ran' and Q for 'ran quickly'.*

Of course any deeper analysis of the adverb 'quickly' would be great.

(iii) Tom opened the box and Tim opened the tin with his pocket knife. 2

Answer. Perhaps: $\exists x\,(Pabx \wedge Pa_1b_1c)$*. But this answer is problematic as Tom might have opened the bow without any tool. If so, then one would be forced to use two different predicate letters: a binary one and a ternary one.*

P: *… opened … with …*
a: *Tom*
b: *the box*
a_1: *Tim*
b_1: *the tin*
c: *Tim's pocket knife*

(iv) Tim is looking for a pen. 2

Answer. Pa *It doesn't say that there is a pen for which Tim is looking.*

(v) Tom finds a pen. 2

Answer. $\exists x\,(Px \wedge Rax)$ *as Tom can only find existing objects.*

P: *… is a pen*
R: *… finds …*

(c) Are the following arguments logically valid? Are they valid in predicate logic (without identity)? Explain your answer.

(i) Tim is a logician. Tom is a logician. Therefore there is something they have in common. 2

Answer. Validity is controversial. I would say that it's not logically valid as it depends on the existence of a shared entity. Not valid in predicate logic with identity unless the sentences are formalised in a very special way.

(ii) Tim is in Birmingham. Hence it's possible that Tim is in Birmingham. 2

Answer. I guess most people would think that this argument is logically valid but one might also take a deviating view. Hardly valid in predicate logic.

(iii) Tim isn't in London. Therefore Tom doesn't know that Tim is in London. 2

Answer. Less likely to be valid that the previous one. Not valid in predicate logic.

(iv) Many students answered question 7 because they thought it was easier than the other questions. Therefore some students answered question 7. 2

Answer. Probably valid, but not valid in predicate logic. Two difficulties: factivity of knowledge and the quantifier 'many'.