I'll prove the following claim:

$$P \to \exists y Ry \vdash \exists y(P \to Ry)$$

This is an unpleasant proof.

I will prove the conclusion by assuming its negation $\neg\exists y(P \to Ry)$ and proving a contradiction. ¬Elim will then allow me to obtain $\exists y(P \to Ry)$.

From $\neg\exists y(P \to Ry)$, which is logically equivalent to $\forall y \neg(P \to Ry)$, one can derive P. Combining this with the premiss gives $\exists y Ry$. If now Rb is assumed this can be shown to contradict $\neg\exists y(P \to Ry)$.

First I'll derive P from $\neg\exists y(P \to Ry)$. I assume P and $\neg P$. I'll try to get a contradiction to conclude P by ¬Elim.

P $\qquad$ $\neg P$

Applying ¬Elim to $\neg Ra$ gives Ra. This is ok even though $\neg Ra$ has never been assumed.

$$\frac{P \qquad \neg P}{Ra}$$

Now I apply →Intro.

$$\dfrac{\dfrac{[P] \qquad \neg P}{Ra}}{P \to Ra}$$

This is an application of ∃Intro.

$$\dfrac{\dfrac{\dfrac{[P] \qquad \neg P}{Ra}}{P \to Ra}}{\exists y(P \to Ry)}$$

Following my plan, I assume $\neg\exists y(P \to Ry)$ to get a contradiction.

$$\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad \neg P}{Ra}}{P \to Ra}}{\exists y(P \to Ry)} \qquad \neg\exists y(P \to Ry)}{}$$

Applying $\neg$Elim yields P.

$$\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad [\neg P]}{Ra}}{P \to Ra}}{\exists y(P \to Ry)} \quad \neg\exists y(P \to Ry)}{P}$$

Now I assume the premiss…

$$\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad [\neg P]}{Ra}}{P \to Ra}}{\exists y(P \to Ry)} \quad \neg\exists y(P \to Ry)}{P} \qquad P \to \exists y Ry$$

to get $\exists y Ry$ by →Elim.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad [\neg P]}{Ra}}{P \to Ra}}{\exists y(P \to Ry)} \quad \neg\exists y(P \to Ry)}{P} \qquad P \to \exists y Ry}{\exists y Ry}$$

To use $\exists y Ry$, I assume Rb, which will be discharged by ∃Elim.

$$\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad [\neg P]}{Ra}}{P \to Ra}}{\exists y(P \to Ry)} \quad \neg\exists y(P \to Ry)}{P} \qquad \qquad$$

$$\dfrac{P \qquad P \to \exists y Ry}{\exists y Ry}$$

$$Rb$$

$P \to \exists y Ry \vdash \exists y(P \to Ry)$

The conclusion is easily derived from this by →Intro and ∃Intro.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad [\neg P]}{Ra}}{P \to Ra}}{\exists y(P \to Ry)} \quad \neg\exists y(P \to Ry)}{P} \qquad P \to \exists y Ry}{\exists y Ry}$$

$$\dfrac{\dfrac{Rb}{P \to Rb}}{\exists y(P \to Ry)}$$

Now I apply ∃Elim.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad [\neg P]}{Ra}}{P \to Ra}}{\exists y(P \to Ry)} \qquad \neg\exists y(P \to Ry)}{P} \qquad P \to \exists y Ry}{\exists y Ry} \qquad \dfrac{\dfrac{[Rb]}{P \to Rb}}{\exists y(P \to Ry)}}{\exists y(P \to Ry)}$$

I have derived the conclusion. The only missing bit is that $\neg\exists y(P \to Ry)$ is still not discharged. So I assume $\neg\exists y(P \to Ry)$ once more.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad [\neg P]}{Ra}}{P \to Ra}}{\exists y(P \to Ry)} \quad \neg\exists y(P \to Ry)}{P} \qquad P \to \exists y Ry}{\exists y Ry} \qquad \dfrac{\dfrac{[Rb]}{P \to Rb}}{\exists y(P \to Ry)}}{\exists y(P \to Ry)} \qquad \neg\exists y(P \to Ry)$$

¬Elim allows me to discharge both assumptions of $\neg\exists y(P \to Ry)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P] \qquad [\neg P]}{Ra}}{P \to Ra}}{\exists y(P \to Ry)} \quad [\neg\exists y(P \to Ry)]}{P} \qquad P \to \exists y Ry}{\exists y Ry} \qquad \dfrac{\dfrac{[Rb]}{P \to Rb}}{\exists y(P \to Ry)}}{\exists y(P \to Ry)} \qquad [\neg\exists y(P \to Ry)]}{\exists y(P \to Ry)}$$