

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

We want to show that  $\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$ .

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

We want to show that  $\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$ .

I will prove the conclusion by proving  $Pa \rightarrow Qa$  and applying  $\forall$ Intro.

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

We want to show that  $\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$ .

I will prove the conclusion by proving  $Pa \rightarrow Qa$  and applying  $\forall$ Intro.

To prove  $Pa \rightarrow Qa$  I assume its negation  $\neg(Pa \rightarrow Qa)$  and prove from this  $Pa \wedge \neg Qa$ . Applying  $\exists$ Intro will give me  $\exists x (Px \wedge \neg Qx)$ , which contradicts the premiss and I can conclude  $Pa \rightarrow Qa$  by applying  $\neg$ Elim.

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

I write down the negation of  
 $Pa \rightarrow Qa$  and try to obtain  $Pa$ .  
To this end I assume  $Pa$  and  
 $\neg Pa$ .

$$\neg Pa$$

$$Pa$$

$$\neg(Pa \rightarrow Qa)$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

Using  $\neg$ -Elim I conclude  $Qa$ .

$$\frac{\begin{array}{c} \neg Pa \\[1ex] Pa \end{array}}{Qa} \quad \neg(Pa \rightarrow Qa)$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

Now I apply  $\rightarrow$ -Intro to  
discharge  $Pa$ .

$$\frac{\begin{array}{c} \neg Pa \\ [Pa] \end{array}}{\frac{Qa}{\frac{Pa \rightarrow Qa}{\neg(Pa \rightarrow Qa)}}}$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

This yields a contradiction,  
which allows me to apply  
 $\neg$ -Elim and  $\neg Pa$  is discharged  
as well.

$$\frac{\begin{array}{c} [\neg Pa] \qquad [Pa] \\ \hline Qa \\ \hline \frac{\begin{array}{c} Pa \rightarrow Qa \qquad \neg(Pa \rightarrow Qa) \\ \hline Pa \end{array}}{Pa} \end{array}}{Pa}$$

$$\neg\exists x(Px \wedge \neg Qx) \vdash \forall x(Px \rightarrow Qx)$$

Following my plan, I try to prove  $\neg Qa$  from the assumption  $\neg(Pa \rightarrow Qa)$ . I try to do this by assuming  $Qa$  hoping that I can apply  $\neg$ -Intro.

$$\frac{\begin{array}{c} [\neg Pa] \qquad [Pa] \\ \hline Qa \\ \hline \frac{\begin{array}{c} Pa \rightarrow Qa \qquad \neg(Pa \rightarrow Qa) \\ \hline Pa \end{array}}{} \end{array}}{}$$

$$\frac{\begin{array}{c} Qa \\ \hline \neg(Pa \rightarrow Qa) \end{array}}{}$$

$$\neg\exists x(Px \wedge \neg Qx) \vdash \forall x(Px \rightarrow Qx)$$

But that's easy because I can  
derive  $Pa \rightarrow Qa$  by  $\rightarrow$ Intro...

$$\frac{\begin{array}{c} [\neg Pa] \qquad [Pa] \\ \hline Qa \\ \hline \frac{Pa \rightarrow Qa \quad \neg(Pa \rightarrow Qa)}{Pa} \end{array}}{\frac{Qa}{Pa \rightarrow Qa} \quad \neg(Pa \rightarrow Qa)}$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

and applying then  $\neg$ -Intro.

$$\frac{\begin{array}{c} [\neg Pa] \quad [Pa] \\ \hline Qa \\ \hline \frac{\begin{array}{c} Pa \rightarrow Qa \quad \neg(Pa \rightarrow Qa) \\ \hline Pa \end{array}}{} \end{array}}{}$$

$$\frac{\begin{array}{c} [Qa] \\ \hline Pa \rightarrow Qa \\ \hline \neg(Pa \rightarrow Qa) \end{array}}{\neg Qa}$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

The plan is to prove  
 $Pa \wedge \neg Qa\dots$

$$\frac{\begin{array}{c} [\neg Pa] \quad [Pa] \\ \hline Qa \\ \hline \frac{\begin{array}{c} Pa \rightarrow Qa \quad \neg(Pa \rightarrow Qa) \\ \hline Pa \end{array}}{\hline} \end{array}}{\hline} \quad \frac{\begin{array}{c} [Qa] \\ \hline Pa \rightarrow Qa \\ \hline \neg(Pa \rightarrow Qa) \\ \hline \neg Qa \end{array}}{\hline}$$
$$\frac{\hline}{Pa \wedge \neg Qa}$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

... and to apply  $\exists$ Intro,

$$\frac{\frac{\frac{[\neg Pa]}{Qa} \quad [Pa]}{\frac{Pa \rightarrow Qa}{\neg(Pa \rightarrow Qa)}} \quad \frac{[Qa]}{Pa \rightarrow Qa}}{\frac{Pa}{\frac{\frac{Pa \wedge \neg Qa}{\exists x (Px \wedge \neg Qx)}}{\neg(Pa \rightarrow Qa)}}}$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

which contradicts the premiss.

$$\frac{\begin{array}{c} [\neg Pa] \quad [Pa] \\ \hline \frac{Qa}{\frac{Pa \rightarrow Qa}{\frac{\neg(Pa \rightarrow Qa)}{Pa}}} \end{array}}{\frac{Pa \wedge \neg Qa}{\exists x (Px \wedge \neg Qx)}} \quad \frac{\begin{array}{c} [Qa] \\ \hline \frac{Pa \rightarrow Qa}{\frac{\neg(Pa \rightarrow Qa)}{\neg Qa}} \end{array}}{\neg \exists x (Px \wedge \neg Qx)}$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

So using  $\neg$ -Elim I infer  
 $Pa \rightarrow Qa$  and discharge  
 $\neg(Pa \rightarrow Qa)$ .

$$\frac{\begin{array}{c} [\neg Pa] \quad [Pa] \\ \hline Qa \\ \hline \frac{\begin{array}{c} Pa \rightarrow Qa \quad [\neg(Pa \rightarrow Qa)] \\ \hline Pa \end{array}}{\frac{\begin{array}{c} Pa \wedge \neg Qa \\ \hline \exists x (Px \wedge \neg Qx) \end{array}}{Pa \rightarrow Qa}} \end{array}}{\neg Qa}$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

Finally there is no  
undischarged assumption left  
that contains the constant  $a$ .  
Thus, I can apply  $\forall$ Intro.

$$\frac{\begin{array}{c} [\neg Pa] \quad [Pa] \\ \hline \frac{\begin{array}{c} Qa \\ \hline \frac{\begin{array}{c} Pa \rightarrow Qa \quad [\neg(Pa \rightarrow Qa)] \\ \hline Pa \end{array}}{\hline \begin{array}{c} Pa \wedge \neg Qa \\ \hline \frac{\begin{array}{c} \exists x (Px \wedge \neg Qx) \quad \neg \exists x (Px \wedge \neg Qx) \\ \hline Pa \rightarrow Qa \end{array}}{\hline \forall x (Px \rightarrow Qx)} \end{array}} \end{array}}{\hline \begin{array}{c} [Qa] \\ \hline \frac{\begin{array}{c} Pa \rightarrow Qa \quad [\neg(Pa \rightarrow Qa)] \\ \hline \neg Qa \end{array}}{\hline \begin{array}{c} Pa \wedge \neg Qa \\ \hline \frac{\begin{array}{c} \exists x (Px \wedge \neg Qx) \quad \neg \exists x (Px \wedge \neg Qx) \\ \hline Pa \rightarrow Qa \end{array}}{\hline \forall x (Px \rightarrow Qx)} \end{array}} \end{array}}$$

$$\neg \exists x (Px \wedge \neg Qx) \vdash \forall x (Px \rightarrow Qx)$$

Apart from the premiss there is no undischarged assumption left. So the proof is complete.

$$\begin{array}{c} [\neg Pa] \quad [Pa] \\ \hline \frac{Qa}{\frac{Pa \rightarrow Qa \quad [\neg(Pa \rightarrow Qa)]}{\frac{Pa}{\frac{\begin{array}{c} Pa \wedge \neg Qa \\ \hline \exists x (Px \wedge \neg Qx) \end{array}}{\frac{\neg \exists x (Px \wedge \neg Qx)}{\frac{Pa \rightarrow Qa}{\frac{\forall x (Px \rightarrow Qx)}{}}}}}}}} \\ \hline \frac{[Qa]}{\frac{Pa \rightarrow Qa \quad [\neg(Pa \rightarrow Qa)]}{\frac{\neg Qa}{}}} \end{array}$$