We want to show that $\neg\exists x\,(Px \land \neg Qx) \vdash \forall x\,(Px \to Qx)$.

We want to show that $\neg\exists x\,(Px \wedge \neg Qx) \vdash \forall x\,(Px \to Qx)$.

I will prove the conclusion by proving $Pa \to Qa$ and applying $\forall$Intro.

We want to show that $\neg\exists x\,(Px \land \neg Qx) \vdash \forall x\,(Px \to Qx)$.

I will prove the conclusion by proving $Pa \to Qa$ and applying $\forall$Intro.

To prove $Pa \to Qa$ I assume its negation $\neg(Pa \to Qa)$ and prove from this $Pa \land \neg Qa$. Applying $\exists$Intro will give me $\exists x\,(Px \land \neg Qx)$, which contradicts the premiss and I can conclude $Pa \to Qa$ by applying $\neg$Elim.

I write down the negation of $Pa \to Qa$ and try to obtain Pa. To this end I assume Pa and $\neg Pa$.

$\neg Pa \qquad Pa$

$$\neg(Pa \to Qa)$$

Using ¬Elim I conclude Qa.

$$\dfrac{\neg Pa \qquad\qquad Pa}{Qa}$$

$$\neg(Pa \to Qa)$$

Now I apply →Intro to discharge *Pa*.

$$\dfrac{\dfrac{\neg Pa \qquad [Pa]}{Qa}}{Pa \to Qa} \qquad \neg(Pa \to Qa)$$

This yields a contradiction, which allows me to apply ¬Elim and $\neg Pa$ is discharged as well.

$$\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \rightarrow Qa} \qquad \neg(Pa \rightarrow Qa)}{Pa}$$

Following my plan, I try to prove $\neg Qa$ from the assumption $\neg(Pa \to Qa)$. I try to do this by assuming Qa hoping that I can apply ¬Intro.

$$\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{Pa}$$

$$Qa \qquad\qquad \neg(Pa \to Qa)$$

But that's easy because I can derive $Pa \to Qa$ by →Intro…

$$\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{Pa}$$

$$\dfrac{Qa}{Pa \to Qa} \qquad \neg(Pa \to Qa)$$

and applying then ¬Intro.

$$\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{Pa} \qquad\qquad \dfrac{\dfrac{[Qa]}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{\neg Qa}$$

The plan is to prove
$Pa \land \neg Qa \ldots$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{Pa} \qquad \dfrac{\dfrac{[Qa]}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{\neg Qa}}{Pa \land \neg Qa}$$

… and to apply ∃Intro,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{Pa} \qquad \dfrac{\dfrac{[Qa]}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{\neg Qa}}{Pa \land \neg Qa}}{\exists x\,(Px \land \neg Qx)}$$

which contradicts the premiss.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{Pa} \qquad \dfrac{\dfrac{[Qa]}{Pa \to Qa} \qquad \neg(Pa \to Qa)}{\neg Qa}}{Pa \wedge \neg Qa}}{\exists x\,(Px \wedge \neg Qx)} \qquad \neg\exists x\,(Px \wedge \neg Qx)$$

So using ¬Elim I infer $Pa \to Qa$ and discharge $\neg(Pa \to Qa)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \to Qa} \quad [\neg(Pa \to Qa)]}{Pa} \qquad \dfrac{\dfrac{[Qa]}{Pa \to Qa} \quad [\neg(Pa \to Qa)]}{\neg Qa}}{Pa \land \neg Qa}}{\exists x\,(Px \land \neg Qx)} \qquad \neg\exists x\,(Px \land \neg Qx)}{Pa \to Qa}$$

Finally there is no undischarged assumption left that contains the constant a. Thus, I can apply $\forall$Intro.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \to Qa} \quad [\neg(Pa \to Qa)]}{Pa} \qquad \dfrac{\dfrac{[Qa]}{Pa \to Qa} \quad [\neg(Pa \to Qa)]}{\neg Qa}}{Pa \wedge \neg Qa}}{\exists x\,(Px \wedge \neg Qx)} \qquad \neg\exists x\,(Px \wedge \neg Qx)}{\dfrac{Pa \to Qa}{\forall x\,(Px \to Qx)}}$$

Apart from the premiss there is no undischarged assumption left. So the proof is complete.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg Pa] \qquad [Pa]}{Qa}}{Pa \to Qa} \quad [\neg(Pa \to Qa)]}{Pa} \qquad \dfrac{\dfrac{[Qa]}{Pa \to Qa} \qquad [\neg(Pa \to Qa)]}{\neg Qa}}{Pa \wedge \neg Qa}}{\exists x\,(Px \wedge \neg Qx)} \qquad \neg\exists x\,(Px \wedge \neg Qx)}{Pa \to Qa}}{\forall x\,(Px \to Qx)}
$$