We want to show that

$$\forall x(Px \to \exists y(Py \land Rxy)), Pa \vdash \exists y \exists z(Ray \land Ryz).$$

This could be the formalisation of the following argument:

> *Every woman has a woman as friend. Tina is a woman. Therefore Tina has a friend who has a friend.*

The strategy for the proof is as follows:

1. Show that Tina has a female friend, ie that there is something that is a woman and a friend of Tina's.
2. Call this friend of Tina's b
3. Show that b has a friend.
4. Call this friend of b's c.
5. Conclude that Tina (ie a) has b as friend and b has c as a friend.
6. Derive the conclusion: Tina has a friend who has a friend.
7. Therefore Tina has a friend who has a friend. This doesn't depends on how we called these friends in the proof.

This list of steps is of course not part of the solution or the proof. It's supposed to illustrate how one sets out the proof strategy.

I write down the premiss $\forall x(Px \to \exists y(Py \wedge Rxy))$ as assumption and....

$$\forall x(Px \to \exists y(Py \wedge Rxy))$$

apply ∀Elim.

$$\frac{\forall x(Px \to \exists y(Py \land Rxy))}{Pa \to \exists y(Py \land Ray)}$$

Pa is the other premiss, which obviously can be used…

$$Pa \quad \frac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pa \to \exists y(Py \wedge Ray)}$$

to obtain $\exists y(Py \wedge Ray)$ by an application of→Elim.

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pa \to \exists y(Py \wedge Ray)}}{\exists y(Py \wedge Ray)}$$

In order to use $\exists y(Py \land Rby)$ later in an application of ∃Elim, I assume $Pb \land Rab$. (Sorry for putting it so far left, but I need to save space.)

$$Pb \land Rab$$

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pa \to \exists y(Py \land Ray)}}{\exists y(Py \land Ray)}$$

I drop Pb by an application of ∧Elim2.

$$\dfrac{Pb \land Rab}{Pb}$$

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pa \to \exists y(Py \land Ray)}}{\exists y(Py \land Ray)}$$

Now I go through the same reasoning as on the left branch but with b instead of a.

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pa \to \exists y(Py \wedge Ray)}}{\exists y(Py \wedge Ray)} \qquad \dfrac{\dfrac{Pb \wedge Rab}{Pb} \qquad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pb \to \exists y(Py \wedge Rby)}}{\exists y(Py \wedge Rby)}$$

This time I assume $Pc \land Rbc$, so I can apply ∃Elim later. I want to prove $\exists y \exists z(Ray \land Ryz)$, which will be obtained by ∃Intro from $\exists z(Ray \land Ryz)$, which in turn is obtained from $Rab \land Rbc$. So I try to prove Rab and Rbc.

$$Pc \land Rbc$$

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pa \to \exists y(Py \land Ray)}}{\exists y(Py \land Ray)} \qquad \dfrac{\dfrac{Pb \land Rab}{Pb} \qquad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pb \to \exists y(Py \land Rby)}}{\exists y(Py \land Rby)}$$

I get Rbc by applying ∧Elim2.

$$\dfrac{Pc \land Rbc}{Rbc}$$

$$\dfrac{Pa \quad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pa \to \exists y(Py \land Ray)}}{\exists y(Py \land Ray)}$$

$$\dfrac{\dfrac{Pb \land Rab}{Pb} \quad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pb \to \exists y(Py \land Rby)}}{\exists y(Py \land Rby)}$$

I assume $Pb \wedge Rab$ once more, but this time I keep Rab and drop Pb. This is an application of $\wedge$Elim2.

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pa \to \exists y(Py \wedge Ray)}}{\exists y(Py \wedge Ray)}$$

$$\dfrac{\dfrac{Pb \wedge Rab}{Pb} \qquad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pb \to \exists y(Py \wedge Rby)}}{\exists y(Py \wedge Rby)} \qquad \dfrac{Pb \wedge Rab}{Rab} \qquad \dfrac{Pc \wedge Rbc}{Rbc}$$

Applying $\land$Intro gives $Rab \land Rbc$..

$$\dfrac{\dfrac{Pb \land Rab}{Rab} \qquad \dfrac{Pc \land Rbc}{Rbc}}{Rab \land Rbc}$$

$$\dfrac{\dfrac{Pb \land Rab}{Pb} \qquad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pb \to \exists y(Py \land Rby)}}{\exists y(Py \land Rby)}$$

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pa \to \exists y(Py \land Ray)}}{\exists y(Py \land Ray)}$$

Following the plan I apply $\exists$Intro to get $\exists z(Rab \land Rbz)$.

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pa \to \exists y(Py \land Ray)}}{\exists y(Py \land Ray)}$$

$$\dfrac{\dfrac{Pb \land Rab}{Pb} \qquad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pb \to \exists y(Py \land Rby)}}{\exists y(Py \land Rby)} \qquad \dfrac{\dfrac{\dfrac{Pb \land Rab}{Rab} \qquad \dfrac{Pc \land Rbc}{Rbc}}{Rab \land Rbc}}{\exists z(Rab \land Rbz)}$$

$\exists z(Rab \land Rbz)$ doesn't contain c. This allows me to apply $\exists$Elim and to discharge the assumption $Pc \land Rbc$. I could have gone directly to the conclusion $\exists y \exists z(Ray \land Ryz)$ and then have applied $\exists$Elim twice.

$$\dfrac{\dfrac{\dfrac{Pb \land Rab}{Pb} \quad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pb \to \exists y(Py \land Rby)}}{\exists y(Py \land Rby)} \quad \dfrac{\dfrac{\dfrac{Pb \land Rab}{Rab} \quad \dfrac{[Pc \land Rbc]}{Rbc}}{Rab \land Rbc}}{\exists z(Rab \land Rbz)}}{\exists z(Rab \land Rbz)}$$

$$\dfrac{Pa \quad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pa \to \exists y(Py \land Ray)}}{\exists y(Py \land Ray)}$$

I apply ∃Intro once more to get the conclusion.

$$\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pa \to \exists y(Py \wedge Ray)}}{\exists y(Py \wedge Ray)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{Pb \wedge Rab}{Pb} \qquad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pb \to \exists y(Py \wedge Rby)}}{\exists y(Py \wedge Rby)} \qquad \dfrac{\dfrac{\dfrac{Pb \wedge Rab}{Rab} \qquad \dfrac{[Pc \wedge Rbc]}{Rbc}}{Rab \wedge Rbc}}{\exists z(Rab \wedge Rbz)}}{\exists z(Rab \wedge Rbz)}}{\exists y \exists z(Ray \wedge Ryz)}$$

Now I still need to discharge the assumption $Pb \land Rab$, which occurs twice. This is another application of ∃Elim.

$$\dfrac{\dfrac{Pa \quad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pa \to \exists y(Py \land Ray)}}{\exists y(Py \land Ray)} \quad \dfrac{\dfrac{\dfrac{\dfrac{[Pb \land Rab]}{Pb} \quad \dfrac{\forall x(Px \to \exists y(Py \land Rxy))}{Pb \to \exists y(Py \land Rby)}}{\exists y(Py \land Rby)} \quad \dfrac{\dfrac{\dfrac{[Pb \land Rab]}{Rab} \quad \dfrac{[Pc \land Rbc]}{Rbc}}{Rab \land Rbc}}{\exists z(Rab \land Rbz)}}{\exists z(Rab \land Rbz)}}{\exists y \exists z(Ray \land Ryz)}}{\exists y \exists z(Ray \land Ryz)}$$

All assumptions other than premisses are discharged, so the proof is complete.

$$\dfrac{\dfrac{Pa \qquad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pa \to \exists y(Py \wedge Ray)}}{\exists y(Py \wedge Ray)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Pb \wedge Rab]}{Pb} \qquad \dfrac{\forall x(Px \to \exists y(Py \wedge Rxy))}{Pb \to \exists y(Py \wedge Rby)}}{\exists y(Py \wedge Rby)} \qquad \dfrac{\dfrac{\dfrac{[Pb \wedge Rab]}{Rab} \qquad \dfrac{[Pc \wedge Rbc]}{Rbc}}{Rab \wedge Rbc}}{\exists z(Rab \wedge Rbz)}}{\exists z(Rab \wedge Rbz)}}{\exists y \exists z(Ray \wedge Ryz)}}{\exists y \exists z(Ray \wedge Ryz)}$$