

It is to be proved that $P \rightarrow Q \vdash \neg P \vee Q$.

One cannot prove one of the disjuncts, that is, $\neg P$ or Q from the assumption $P \rightarrow Q$. Thus proving first $\neg P$ or Q and then applying \vee Intro1 or \vee Intro2 will hardly work.

In order to prove a disjunction it's often better to assume the negation of the disjunction, in the present case $\neg(\neg P \vee Q)$, and to derive a contradiction, so that one can apply \neg Elim and conclude $\neg P \vee Q$. This is the strategy in the following proof.

I assume P and combine it with the
premiss $P \rightarrow Q$

P $P \rightarrow Q$

By \rightarrow Elim I conclude Q .

$$\frac{P \quad P \rightarrow Q}{Q}$$

Applying \vee Intro2 yields $\neg P \vee Q$.

$$\frac{P \quad P \rightarrow Q}{\frac{Q}{\neg P \vee Q}}$$

I assume the negation of the conclusion...

$$\frac{\frac{P \quad P \rightarrow Q}{Q}}{\neg P \vee Q} \quad \neg(\neg P \vee Q)$$

to get $\neg P$ by \neg -Intro. The assumption P is discharged accordingly.

$$\frac{\frac{[P] \quad P \rightarrow Q}{Q}}{\neg P \vee Q} \quad \neg(\neg P \vee Q)}{\neg P}$$

I apply \vee Intro1...

$$\frac{\frac{[P] \quad P \rightarrow Q}{Q}}{\neg P \vee Q} \quad \neg(\neg P \vee Q)}{\frac{\neg P}{\neg P \vee Q}}$$

and assume once more the negation of the conclusion.

$$\frac{\frac{[P] \quad P \rightarrow Q}{Q}}{\frac{\neg P \vee Q}{\frac{\neg P}{\neg P \vee Q}} \quad \neg(\neg P \vee Q)}}{\neg(\neg P \vee Q)}$$

I derive the conclusion by \neg -Elim.
 $\neg(\neg P \vee Q)$ has been assumed
twice. Both occurrences can be dis-
charged.

$$\frac{\frac{[P] \quad P \rightarrow Q}{Q}}{\frac{\neg P \vee Q}{\frac{\neg P}{\neg P \vee Q} \quad [\neg(\neg P \vee Q)]}}{\frac{\neg P \vee Q}{\frac{\neg P \vee Q}{\neg P \vee Q} \quad [\neg(\neg P \vee Q)]}}$$