I want to show that $\neg(P \to Q) \vdash P \wedge \neg Q$.

At first I explain how to find the proof.

The conclusion is a conjunction. I try to get it by $\wedge$Intro. So the end of the proof looks like this:

$$\dfrac{\dfrac{}{P} \qquad \dfrac{}{\neg Q}}{P \wedge \neg Q}$$

First I turn to the left branch ending with P. I assume $\neg P$. From this and the premise $\neg(P \to Q)$ I try to get a contradiction, so $\neg$Elim can be applied to get P.

$$\dfrac{\dfrac{\neg(P \to Q) \qquad \neg P}{P} \qquad \dfrac{}{\neg Q}}{P \wedge \neg Q}$$

I try to obtain $P \to Q$ from the assumption P by $\to$Intro. This would contradict $\neg(P \to Q)$. So I assume P.

$$P \qquad\qquad \neg P$$

$$\dfrac{\dfrac{\neg(P \to Q) \qquad \dfrac{}{P \to Q}}{P} \qquad \dfrac{}{\neg Q}}{P \land \neg Q}$$

P and $\neg P$ yield already Q by $\neg$Elim (although $\neg Q$ has never been assumed). The assumption P is discharged according to $\to$Intro. The assumption $\neg P$ is discharged according to $\neg$Elim.

$$\dfrac{\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P} \qquad \dfrac{}{\neg Q}}{P \wedge \neg Q}$$

To obtain $\neg Q$ I assume Q in order apply $\neg$Intro. I have discharged Q here already according to $\neg$Intro.

$$\dfrac{\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P} \qquad \dfrac{[Q]}{\neg Q}}{P \land \neg Q}$$

I have the premise $\neg(P \to Q)$ that I can use to produce a contradiction.

$$\dfrac{\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P} \qquad \dfrac{[Q] \qquad \neg(P \to Q)}{\neg Q}}{P \wedge \neg Q}$$

Applying →Intro to Q gives $P \to Q$ though P hasn't been assumed.

$$\dfrac{\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P} \qquad \dfrac{\dfrac{[Q]}{P \to Q} \qquad \neg(P \to Q)}{\neg Q}}{P \wedge \neg Q}$$

Now I go through the proof once more, but this time in the correct order and noting the rules used.

assumptions

P $\neg P$

$\neg$Elim applied to the assumption $\neg Q$. This is again a case where I apply $\neg$Elim to an assumption that has never been made in the proof.

$$\frac{P \qquad \neg P}{Q}$$

$\to$Intro; assumption P discharged

$$\dfrac{\dfrac{[P] \qquad \neg P}{Q}}{P \to Q}$$

premiss

$$\neg(P \to Q) \qquad \cfrac{\cfrac{[P] \qquad \neg P}{Q}}{P \to Q}$$

¬Elim; assumption $\neg P$ discharged

$$\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P}$$

assumption

$$\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P}$$

Q

→Intro

$$\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P} \qquad\qquad \dfrac{Q}{P \to Q}$$

premiss

$$\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P} \qquad \dfrac{Q}{P \to Q} \qquad \neg(P \to Q)$$

¬Intro; assumption Q discharged

$$\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P} \qquad\qquad \dfrac{\dfrac{[Q]}{P \to Q} \qquad \neg(P \to Q)}{\neg Q}$$

$\wedge$Intro

$$\dfrac{\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P} \qquad \dfrac{\dfrac{[Q]}{P \to Q} \qquad \neg(P \to Q)}{\neg Q}}{P \wedge \neg Q}$$

All assumptions are discharged; only the premiss $\neg(P \to Q)$ remains undischarged. Thus the proof is complete.

$$\dfrac{\dfrac{\neg(P \to Q) \qquad \dfrac{\dfrac{[P] \qquad [\neg P]}{Q}}{P \to Q}}{P} \qquad \dfrac{\dfrac{[Q]}{P \to Q} \qquad \neg(P \to Q)}{\neg Q}}{P \wedge \neg Q}$$