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For the Natural Deduction proof I assume $Pa$ (The plan is to apply ∃Elim later using the assumption $\exists x\, Px$). Then I derive from $Pa$ and the premiss a contradiction. Then I assume $\exists x\, Px$ and discharge $Pa$ by applying ∃Elim. By applying ¬Intro I conclude $\neg \exists x\, Px$.

$$\forall x \forall y (Px \to (Py \to \neg x = y))$$

This is the premiss.

$$\cfrac{\cfrac{\forall x\, \forall y\, (Px \to (Py \to \neg x = y))}{\forall y\, (Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}$$

I apply $\forall$Elim twice using the same constant.

$$\dfrac{\dfrac{\forall x\, \forall y\, (Px \to (Py \to \neg x = y))}{\forall y\, (Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)} \qquad Pa$$

Later I will assume $\exists x\, Px$ and show that this leads to a contradiction. So I assume $Pa$…

$$\dfrac{Pa \qquad \dfrac{\dfrac{\forall x\, \forall y\, (Px \to (Py \to \neg x = y))}{\forall y\, (Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}}{Pa \to \neg a = a}$$

and apply →Elim.

$$
\dfrac{Pa \qquad \dfrac{Pa \qquad \dfrac{\dfrac{\forall x \, \forall y \, (Px \to (Py \to \neg x = y))}{\forall y \, (Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}}{Pa \to \neg a = a}}{}
$$

This is repeated,

$$\dfrac{Pa \qquad \dfrac{Pa \qquad \dfrac{\dfrac{\forall x\, \forall y\, (Px \to (Py \to \neg x = y))}{\forall y\, (Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}}{Pa \to \neg a = a}}{\neg a = a}$$

which gives me $\neg a = a$

$$
[a = a] \qquad \dfrac{Pa \qquad \dfrac{Pa \qquad \dfrac{\dfrac{\forall x \, \forall y \, (Px \to (Py \to \neg x = y))}{\forall y \, (Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}}{Pa \to \neg a = a}}{\neg a = a}
$$

This is an application of =Intro.

$$\dfrac{[a = a] \qquad \dfrac{Pa \qquad \dfrac{Pa \qquad \dfrac{\dfrac{\forall x\,\forall y\,(Px \to (Py \to \neg x = y))}{\forall y\,(Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}}{Pa \to \neg a = a}}{\neg a = a}}{\neg\exists x\, Px}$$

I apply ¬Intro to $\exists x\, Px$. I can’t discharge any assumption because I didn’t assume $\exists x\, Px$.

$$\exists x\,Px \qquad \dfrac{[a = a] \qquad \dfrac{Pa \qquad \dfrac{Pa \qquad \dfrac{\dfrac{\forall x\,\forall y\,(Px \to (Py \to \neg x = y))}{\forall y\,(Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}}{Pa \to \neg a = a}}{\neg a = a}}{\neg\exists x\,Px}$$

Following my plan, I assume $\exists x\,Px$.

$$\dfrac{\exists x\, Px \qquad \dfrac{[a = a] \qquad \dfrac{[Pa] \qquad \dfrac{[Pa] \qquad \dfrac{\dfrac{\forall x\,\forall y\,(Px \to (Py \to \neg x = y))}{\forall y\,(Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}}{Pa \to \neg a = a}}{\neg a = a}}{\neg\exists x\, Px}}{\neg\exists x\, Px}$$

This is a correct application of ∃Elim because the only undischarged assumption containing $a$ is $Pa$ and $\neg\exists x\, Px$ doesn't contain $a$. So I can discharge the assumptions of $Pa$.

$$\dfrac{\exists x\,Px \qquad \dfrac{[a = a] \qquad \dfrac{[Pa] \qquad \dfrac{[Pa] \qquad \dfrac{\dfrac{\forall x\,\forall y\,(Px \to (Py \to \neg x = y))}{\forall y\,(Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}}{Pa \to \neg a = a}}{\neg a = a}}{\neg\exists x\,Px}}{\neg\exists x\,Px} \qquad\qquad \exists x\,Px$$

I still need to discharge the assumption $\exists x\,Px$. To this end I assume it and obtain a contradiction

$$
\dfrac{\dfrac{[\exists x\,Px] \qquad \dfrac{[a = a] \qquad \dfrac{[Pa] \qquad \dfrac{[Pa] \qquad \dfrac{\dfrac{\forall x\,\forall y\,(Px \to (Py \to \neg x = y))}{\forall y\,(Pa \to (Py \to \neg a = y))}}{Pa \to (Pa \to \neg a = a)}}{Pa \to \neg a = a}}{\neg a = a}}{\neg\exists x\,Px}}{\neg\exists x\,Px} \qquad [\exists x\,Px]}{\neg\exists x\,Px}
$$

Using ¬Intro I discharge both assumptions of $\exists x\,Px$ and the proof is complete..