I will give a proof for the following claim:

$$Pa, Pb \vdash \forall x(x = a \lor x = b \rightarrow Px)$$

I'll first show how to arrive at the proof. Then I'll go through the proof step-by-step according to the rules.

Assuming the premisses is usually a good idea.

Pa

Pb

I need to prove the conclusion $\forall x (x = a \lor x = b \rightarrow Px)$ from these premisses.



Pb

$$\forall x (x = a \lor x = b \to Px)$$

I hope to obtain the conclusion by an application of \forall Intro.

Pb

$$\frac{c = a \lor c = b \to Pc}{\forall x (x = a \lor x = b \to Px)}$$

 $c = a \lor c = b \rightarrow Pc$ will be arrived at by \rightarrow Intro. So I assume $c = a \lor c = b$ and try to get *Pc* from it and from the other two assumptions.



Pb

 $c = a \lor c = b$

$$\frac{c = a \lor c = b \to Pc}{\forall x (x = a \lor x = b \to Px)}$$

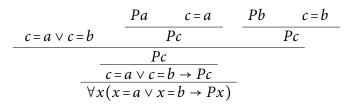
In order to derive something from $c = a \lor c = b$, I use \lor Elim. So I assume c = a and c = b.

$$Pa \qquad c=a \qquad Pb \qquad c=b$$

$$c=a \lor c=b$$

$$\frac{c = a \lor c = b \to Pc}{\forall x (x = a \lor x = b \to Px)}$$

Obtaining Pc by =Elim is now easy. Since Pc can be obtained in both cases, \lor Elim can be applied.



...and yes the square brackets are still missing: $c = a \lor c = b$ is discharged according to \rightarrow Intro and c = a and c = b are discharged according to \lor Elim.

$$\begin{array}{c|c}
Pa & [c=a] \\
\hline Pc & Pc \\
\hline \hline \hline c=a \lor c=b \to Pc \\
\hline \forall x(x=a \lor x=b \to Px) \\
\end{array}$$

To make sure that everything is correct, I go through the proof in the offical order.

Pa and Pb, c = a and c = b are assumed.

$$Pa$$
 $c=a$ Pb $c=b$

=Elim is applied in both cases to get *Pc*.

$$\frac{Pa \qquad c=a}{Pc} \qquad \frac{Pb \qquad c=b}{Pc}$$

$c = a \lor c = b$ is assumed.

$$c = a \lor c = b \qquad \frac{Pa \qquad c = a}{Pc} \qquad \frac{Pb \qquad c = b}{Pc}$$

_

 \lor Elim can be applied and c = a and c = b are discharged in accordance with \lor Elim.

$$\begin{array}{c|c} Pa & [c=a] \\ \hline c=a \lor c=b \\ \hline Pc \\ \hline Pc \\ \hline \end{array} \begin{array}{c} Pb & [c=b] \\ \hline Pc \\ \hline \end{array}$$

Now \rightarrow Intro is used to discharge $c = a \lor c = b$ and to get $c = a \lor c = b \rightarrow Pc$.

$$\frac{Pa \quad [c=a]}{Pc} \quad \frac{Pb \quad [c=b]}{Pc} \\
 \frac{Pc}{c=a \lor c=b \to Pc}$$

The last step is an application of \forall Intro. All assumptions containing *c* have already been discharged and the conclusion $\forall x (x = a \lor x = b \Rightarrow Px)$ also does not contain *c*; so the condition on constants is satisfied.

$$\frac{Pa \quad [c=a]}{Pc} \quad \frac{Pb \quad [c=b]}{Pc} \\
 \frac{Pb \quad [c=b]}{Pc} \\
 \frac{Pc}{c=a \lor c=b \to Pc} \\
 \forall x(x=a \lor x=b \to Px)$$