

I will give a proof for the following claim:

$$Pa, Pb \vdash \forall x(x = a \vee x = b \rightarrow Px)$$

I'll first show how to arrive at the proof. Then I'll go through the proof step-by-step according to the rules.

Assuming the premisses is usually  
a good idea.

*Pa*

*Pb*

I need to prove the conclusion  $\forall x(x = a \vee x = b \rightarrow P x)$  from these premisses.

 $P a$  $P b$ 

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 $\forall x(x = a \vee x = b \rightarrow P x)$

I hope to obtain the conclusion by  
an application of  $\forall$ Intro.

$P a$

$P b$

$$\frac{c = a \vee c = b \rightarrow P c}{\forall x(x = a \vee x = b \rightarrow P x)}$$

$c = a \vee c = b \rightarrow P c$  will be arrived at by  $\rightarrow$ Intro. So I assume  $c = a \vee c = b$  and try to get  $P c$  from it and from the other two assumptions.

$$\begin{array}{ccc} & P a & P b \\ c = a \vee c = b & & \\ \hline \frac{c = a \vee c = b \rightarrow P c}{\forall x(x = a \vee x = b \rightarrow P x)} & & \end{array}$$

In order to derive something from  $c = a \vee c = b$ , I use  $\vee$ Elim. So I assume  $c = a$  and  $c = b$ .

$$\begin{array}{cccc} & P a & c = a & P b & c = b \\ c = a \vee c = b & & & & \\ & \overline{c = a \vee c = b \rightarrow P c} & & & \\ & \forall x(x = a \vee x = b \rightarrow P x) & & & \end{array}$$

Obtaining  $P c$  by =Elim is now easy.  
Since  $P c$  can be obtained in both cases,  $\vee$ Elim can be applied.

$$\frac{c = a \vee c = b \quad \frac{P a \quad c = a}{P c} \quad \frac{P b \quad c = b}{P c}}{P c}$$
$$\frac{c = a \vee c = b \rightarrow P c}{\forall x(x = a \vee x = b \rightarrow P x)}$$

...and yes the square brackets are still missing:  $c = a \vee c = b$  is discharged according to  $\rightarrow$ Intro and  $c = a$  and  $c = b$  are discharged according to  $\vee$ Elim.

$$\frac{\frac{[c = a \vee c = b] \quad \frac{P a \quad [c = a]}{P c} \quad \frac{P b \quad [c = b]}{P c}}{P c}}{c = a \vee c = b \rightarrow P c}}{\forall x(x = a \vee x = b \rightarrow P x)}$$



To make sure that everything is correct, I go through the proof in the official order.

$P a$  and  $P b$ ,  $c = a$  and  $c = b$  are assumed.

$P a$        $c = a$        $P b$        $c = b$

=Elim is applied in both cases to get  $P c$ .

$$\frac{P a \quad c = a}{P c} \quad \frac{P b \quad c = b}{P c}$$

$c = a \vee c = b$  is assumed.

$$c = a \vee c = b \quad \frac{P a \quad c = a}{P c} \quad \frac{P b \quad c = b}{P c}$$

$\forall$ Elim can be applied and  $c = a$  and  $c = b$  are discharged in accordance with  $\forall$ Elim.

$$\frac{c = a \vee c = b \quad \frac{P a \quad [c = a]}{P c} \quad \frac{P b \quad [c = b]}{P c}}{P c}$$

Now  $\rightarrow$ Intro is used to discharge  $c = a \vee c = b$  and to get  $c = a \vee c = b \rightarrow P c$ .

$$\frac{\frac{[c = a \vee c = b] \quad \frac{P a \quad [c = a]}{P c} \quad \frac{P b \quad [c = b]}{P c}}{P c}}{c = a \vee c = b \rightarrow P c}$$

The last step is an application of  $\forall$ Intro. All assumptions containing  $c$  have already been discharged and the conclusion  $\forall x(x = a \vee x = b \rightarrow P x)$  also does not contain  $c$ ; so the condition on constants is satisfied.

$$\frac{\frac{[c = a \vee c = b] \quad \frac{\frac{P a \quad [c = a]}{P c} \quad \frac{P b \quad [c = b]}{P c}}{P c}}{c = a \vee c = b \rightarrow P c}}{\forall x(x = a \vee x = b \rightarrow P x)}$$