INTRODUCTION TO LOGIC

Lecture 7
Formalisation in Predicate Logic

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'Contrariwise,' continued Tweedledee,
'if it was so, it might be;
and if it were so, it would be;
but as it isn't, it ain't.
That's logic'
Lewis Carroll
Through the Looking-Glass

Recap: Adequacy

Two notions of consequence coincide.

Let Γ be a set of $\mathcal{L}_2$-sentences and ϕ a $\mathcal{L}_2$-sentence

Definition: provable (syntactic)

$\Gamma \vdash \phi$ iff there is a proof of ϕ with only sentences in Γ as non-discharged assumptions.

Definition: valid (semantic)

$\Gamma \models \phi$ iff there is no $\mathcal{L}_2$-structure in which all sentences in Γ are true and ϕ is false.

Adequacy theorem (Soundness and Completeness)

$\Gamma \vdash \phi$ iff $\Gamma \models \phi$

How does validity in $\mathcal{L}_2$ relate to English?

Definition: valid in predicate logic

An argument in English is *valid in predicate logic* iff its formalisation in the language $\mathcal{L}_2$ is valid.

Suppose we have an English argument.

Method to establish validity

Step (i) Formalise the argument in $\mathcal{L}_2$.
Step (ii) Prove the formalised argument in Natural Deduction.

Method to establish non-validity

Step (i) Formalise the argument in $\mathcal{L}_2$.
Step (ii) Construct a counterexample. (An $\mathcal{L}_2$-structure in which the premisses are true and the conclusion is false.)

Unfinished business

Show that argument 2 is valid

(1) Zeno is a tortoise.
(2) All tortoises are toothless.
Therefore, (C) Zeno is toothless.

Step (i): formalise

(1) Ta
(2) $\forall x(Tx \to Lx)$
(C) La

Dictionary: a: Zeno. T:. . . is a tortoise. L: . . . is toothless

Step (ii): show the formalisation is valid

Need to show: $Ta, \forall x(Tx \to Lx) \models La$.
Sufficient to show: $Ta, \forall x(Tx \to Lx) \vdash La$.

Sufficient to show: $Ta, \forall x(Tx \to Lx) \vdash La$.

$$\dfrac{Ta \qquad \dfrac{\forall x(Tx \to Lx)}{Ta \to La}}{La}$$

That completes the proof.
Consequently: $Ta, \forall x(Tx \to Lx) \models La$.
The English argument about Zeno is valid in predicate logic.

Exercise: show that the following argument is valid

All concrete objects are located in space. The number 5 isn't located in space. So the number 5 isn't a concrete object.

Dictionary: a: the number 5.
C: … is a concrete object. L: … is located in space.

formalisation

Premiss 1: $\forall x\,(Cx \to Lx)$
Premiss 2: $\neg La$
Conclusion: $\neg Ca$

$$\dfrac{\dfrac{[Ca] \qquad \dfrac{\forall x(Cx \to Lx)}{Ca \to La}}{La} \qquad \neg La}{\neg Ca}$$

So the argument is valid in predicate logic.

Note on partial formalisation

To establish validity: partial formalisation may suffice.

	Less detailed (I)	More detailed (II)	Not detailed enough (III)
Premiss 1	$\forall x\,(Cx \to Lx)$	$\forall x\,(Cx \to L^2xb)$	A
Premiss 2	$\neg La$	$\neg Lab$	S
Conclusion	$\neg Ca$	$\neg Ca$	C

Both (I) and (II) are fine. (III) is not.

Dictionary: b: Space. L^2: … is located in … .

NB: to show non-validity: full formalisation required.

A: All concrete objects are located in space.
S: The number 5 isn't located in space.
C The number 5 isn't a concrete object.

Example: show the following argument is not valid

A belief is known only if it is true and justified. The belief that Jones is in Barcelona or Jones owns a Ford is true and justified. Therefore, it's known.

Step (i): formalise

Step (ii): construct a counterexample.

Other logical notions

Let ϕ be an $\mathcal{L}_2$-sentence:

ϕ is a logical truth iff:

- ϕ is true in every $\mathcal{L}_2$-structure. (by definition)
- i.e. $\models \phi$
- i.e. $\vdash \phi$ (Adequacy: $\Gamma \models \phi$ iff $\Gamma \vdash \phi$.)
- i.e. there is a proof of ϕ with no undischarged assumptions.

ϕ is a contradiction iff:

- ϕ is true in no $\mathcal{L}_2$-structure. (by definition)
- i.e. $\neg\phi$ is true in every $\mathcal{L}_2$-structure
- i.e. $\models \neg\phi$
- i.e. $\vdash \neg\phi$
- i.e. there is a proof of $\neg\phi$ with no undischarged assumptions.

Definitions

An English sentence is:

- *logically true in predicate logic* iff its formalisation in predicate logic is logically true.
- a *contradiction in predicate logic* iff its formalisation in predicate logic is a contradiction.

Methods in predicate logic

To show that an English sentence is:

- *logically true* in predicate logic:

Step (i) formalise the sentence as a sentence ϕ of $\mathcal{L}_2$.
Step (ii) prove that $\vdash \phi$.

- *a contradiction* in predicate logic:

Step (i) formalise the sentence as a sentence ϕ of $\mathcal{L}_2$.
Step (ii) prove that $\vdash \neg\phi$.

Example: show the sentence is a contradiction.

Something is bigger than everything but not bigger than itself.

Step (i): formalise

Step (ii): prove the negation of the formalisation

Further issues: scope ambiguity

Every philosopher knows a metaphysician

Paraphrases

(1) Every philosopher is such that they know some metaphysician. Every x is such that (if x is a philosopher, then some y is such that (y is a metaphysician and x knows y)).

(2) Some metaphysician is such that every philosopher knows them. Some y is such that (y is a metaphysician and every x is such that (if x is a philosopher, then x knows y)).

Formalisations

(1) $\forall x(Px \to \exists y(My \wedge Kxy))$

(2) $\exists y(My \wedge \forall x(Px \to Kxy))$

Dictionary: P: ...is a philosopher.
M: ...is a metaphysician. K: ...knows ....

Issue 2: variable arity.

Example: formalise in $\mathcal{L}_2$ as a valid argument.

Manny is eating the scrambled egg out of his shoe with his comb. So, Manny is eating the scrambled egg and Manny is eating out of his shoe.

Clearly the following formalisation is not valid.

Premiss: E_1mesc. Conclusion: $E_2me \wedge E_3ms$.

Dictionary:
m: Manny. e: the scrambled egg.
s: Manny's shoe. c: Manny's comb.
E_1: ...is eating ...out of ...with ...
E_2: ...is eating ....
E_3: ...is eating out of ...

Formalise in $\mathcal{L}_2$ as a valid argument.

Manny is eating the scrambled egg out of his shoe with his comb. So, Manny is eating the scrambled egg and Manny is eating out of his shoe.

The trick is to formalise the argument just using E_1.

Paraphrase of conclusion

Manny is eating the scrambled egg out of something with something and Manny is eating something out of his shoe with something.

Premiss: E_1mesc. Conclusion: $\exists z\exists wE_1mezw \wedge \exists y\exists wE_1mysw$

Dictionary:
m: Manny. e: the scrambled egg.
s: Manny's shoe. c: Manny's comb.
E_1: ...is eating ...out of ...with ...

Issue 3: adverbs

Example: formalise in $\mathcal{L}_2$ as a valid argument.

Usain ran quickly; so Usain ran.

The following formalisation is clearly not valid.

Premiss: Qb. Conclusion: Rb.

Dictionary: b: Usain. Q: ...ran quickly. R: ...ran.

But there is a (somewhat contrived) way to formalise it.

Dictionary: b: Usain. R_1: ...was a running (event).
Q_1: ...was quick. P: ...is the person who did ....

The following is valid:

Premiss: $\exists x(R_1x \wedge Pbx \wedge Q_1x)$. Conclusion: $\exists x(R_1x \wedge Pbx)$.

Issue 4: non-extensionality

Example **Not valid**

Miles wants to live in Oxford. Oxford is a city with high levels of air pollution. Therefore Miles wants to live in a city with high levels of air pollution.

Yet the obvious $\mathcal{L}_2$-formalisation is valid.

Premiss 1: Lmo.
Premiss 2: Po.
Conclusion: $\exists x\,(Lmx \wedge Px)$.

L:	... wants to live in ...
P:	... is a city with high levels of air pollution
m:	Miles
o:	Oxford

Example **Not valid**

Miles wants to live in Oxford. Oxford is a city with high levels of air pollution. Therefore Miles wants to live in a city with high levels of air pollution.

If the formalisation were correct the argument would be valid in predicate logic as

$Lmo, Po \vdash \exists x\,(Lmx \wedge Px)$

$$\dfrac{\dfrac{Lmo \qquad Po}{Lmo \wedge Po}}{\exists x\,(Lmx \wedge Px)}$$

What has gone wrong?

Extensionality of $\mathcal{L}_2$

$\mathcal{L}_2$-structures assign extensions to expressions.

$\mathcal{L}_2$-expression	*extension*
constant	object
sentence	truth-value
unary predicate	set
binary predicate	set of pairs

They have the following feature.

Extensionality

In a $\mathcal{L}_2$-structure, the extension of a sentence depends only on the extensions of its constituent expressions.

Let $\mathcal{A}$ be an $\mathcal{L}_2$-structure, ϕ an $\mathcal{L}_2$-sentence.

Extensionality in $\mathcal{L}_2$

Replacing an expression in ϕ for another with the same extension in $\mathcal{A}$ leaves the extension (truth-value) of ϕ in $\mathcal{A}$ unchanged.

'Coextensive expressions are always substitutable salva veritate'

Examples

(i) Suppose $|Pa|_{\mathcal{A}} = T$ and $|a|_{\mathcal{A}} = |b|_{\mathcal{A}}$. Then $|Pb|_{\mathcal{A}} = T$

(ii) Suppose $|Pa|_{\mathcal{A}} = T$ and $|P|_{\mathcal{A}} = |Q|_{\mathcal{A}}$. Then $|Qa|_{\mathcal{A}} = T$

Proof:

(i) $|Pa|_{\mathcal{A}} = T$; so $|a|_{\mathcal{A}} \in |P|_{\mathcal{A}}$; so $|b|_{\mathcal{A}} \in |P|_{\mathcal{A}}$; so $|Pb|_{\mathcal{A}} = T$
(ii) $|Pa|_{\mathcal{A}} = T$; so $|a|_{\mathcal{A}} \in |P|_{\mathcal{A}}$; so $|a|_{\mathcal{A}} \in |Q|_{\mathcal{A}}$; so $|Qa|_{\mathcal{A}} = T$

We can likewise assign extensions to English expressions.

English expression	*extension*
designator	object
sentence	truth-value

However, some English predicates are non-extensional: we cannot substitute coextensive designators in them without changing the truth-value of the sentence.
Write $|e|$ for the extension of an English expression e.

- |'Superman'| = |'Clark Kent'|
  |'Lois knows that Superman wears a cape'| = T
  |'Lois knows that Clark Kent wears a cape'| = F
- |'8'| = |'the number of Planets'|
  |'If Mars blew up, the number of planets would be 7'| = T
  |'If Mars blew up, 8 would be 7'| = F

Non-extensional predicates

- Miles wants to live in ...
- ... knows that ... wears a cape
- If Mars blew up ... would be ...

We can only formalise extensional predicates as predicate letters in $\mathcal{L}_2$.

Example: formalise in as much detail as possible

Lois knows that Superman wears a cape

Formalisation: Ka
Dictionary: a: Lois.
K: ... knows that Superman wears a cape

'... knows that Superman wears a cape' *is* extensional.