INTRODUCTION TO LOGIC

Lecture 6

Natural Deduction

Dr. James Studd

There's nothing you can't prove
if your outlook is only sufficiently limited
Dorothy L. Sayers

Outline

1. Proof
2. Rules for connectives
3. Rules for quantifiers
4. Adequacy

Proofs in Natural Deduction

- Proofs in Natural Deduction are trees of $\mathcal{L}_2$-sentences

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- The unbracketed sentences at the top are the premisses
- Each line is an instance of one of 17 rules
- The rules depend purely on the syntax of the sentences

  ...not on their semantic properties.

Rules for $\wedge$

$\wedge$Intro

The result of appending $\phi \wedge \psi$ *to a proof of* ϕ *and a proof of* ψ *is a proof of* $\phi \wedge \psi$.

Rules for $\wedge$

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$$\dfrac{\begin{matrix}\vdots & \qquad & \vdots \\ \phi & & \psi\end{matrix}}{\phi \wedge \psi}\ \wedge\text{Intro}$$

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$\wedge$Elim1 and $\wedge$Elim2

(1) *The result of appending ϕ to a proof of $\phi \wedge \psi$ is a proof of ϕ.*

(2) *The result of appending ψ to a proof of $\phi \wedge \psi$ is a proof of ψ.*

$$\dfrac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\phi}\ \wedge\mathit{Elim1} \qquad \dfrac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\psi}\ \wedge\mathit{Elim2}$$

Example

$(P \wedge Q) \wedge R \vdash P$

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$(P \wedge Q) \wedge R \vdash P$

$$(P \wedge Q) \wedge R$$

First, assume the premiss.
This is covered by the

ASSUMPTION RULE

The occurrence of a sentence ϕ with no sentence above it is an assumption. An assumption of ϕ is a proof of ϕ.

You may assume any sentence.
(But choosing the right assumptions is important.)

Example

$(P \wedge Q) \wedge R \vdash P$

$$\frac{(P \wedge Q) \wedge R}{P \wedge Q}$$

Next apply a rule ∧Elim1.

$$\cfrac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\phi}\ \wedge\textit{Elim1}$$

Example

$(P \wedge Q) \wedge R \vdash P$

$$\dfrac{\dfrac{(P \wedge Q) \wedge R}{P \wedge Q}}{P}$$

Next apply the same rule a second time.

$$\dfrac{\begin{array}{c}\vdots \\ \phi \wedge \psi\end{array}}{\phi} \wedge \mathit{Elim1}$$

Example

$(P \wedge Q) \wedge R \vdash P$

That's it! We have a complete proof.

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- The conclusion is the sentence at the root.
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- Each line is a correct application of a Natural Deduction rule.

Example

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

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$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ \phi & \psi \end{array}}{\phi \wedge \psi} \wedge\text{Intro}$$

$$\dfrac{\begin{array}{c} \vdots \\ \phi \wedge \psi \end{array}}{\phi} \wedge \mathit{Elim1} \qquad \dfrac{\begin{array}{c} \vdots \\ \phi \wedge \psi \end{array}}{\psi} \wedge \mathit{Elim2}$$

Example

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

First assume the first premiss.

$$Qb \wedge Pa$$

Example

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

$$\frac{Qb \wedge Pa}{Pa}$$

Next apply a rule for $\wedge$.

$$\begin{array}{c}\vdots \\ \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathit{Elim2}\end{array}$$

Example

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

Now assume the second premiss

$$\dfrac{Qb \wedge Pa}{Pa} \qquad Ra$$

Example

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

$$\dfrac{\dfrac{Qb \wedge Pa}{Pa} \qquad Ra}{Pa \wedge Ra}$$

Now apply the introduction rule for $\wedge$.

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ \phi & & \psi\end{matrix}}{\phi \wedge \psi}\ \wedge\text{Intro}$$

Example

$Qb \wedge Pa, Ra \vdash Pa \wedge Ra$

The proof is complete.

$$\dfrac{\dfrac{Qb \wedge Pa}{Pa} \qquad Ra}{Pa \wedge Ra}$$

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$$\dfrac{\begin{array}{c}\vdots \\ \phi\end{array} \qquad \begin{array}{c}\vdots \\ \phi \rightarrow \psi\end{array}}{\psi}\rightarrow\text{Elim}$$

This rule is often called ‘Modus Ponens’.

Example

$\exists y\, Py \to Qa, \exists y\, Py \vdash Qa$

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$$\dfrac{\begin{matrix}\vdots \\ \phi\end{matrix} \qquad \begin{matrix}\vdots \\ \phi \to \psi\end{matrix}}{\psi} \to\text{Elim}$$

Example

$\exists y\, Py \to Qa, \exists y\, Py \vdash Qa$

Assume both premisses.

$$\exists y\, Py \qquad \exists y\, Py \to Qa$$

Example

$\exists y\, Py \to Qa, \exists y\, Py \vdash Qa$

$$\frac{\exists y\, Py \qquad \exists y\, Py \to Qa}{Qa}$$

Apply the elimination rule.

$$\frac{\begin{matrix}\vdots \\ \phi\end{matrix} \qquad \begin{matrix}\vdots \\ \phi \to \psi\end{matrix}}{\psi} \to\text{Elim}$$

Example

$\exists y\, Py \to Qa, \exists y\, Py \vdash Qa$

Finished!

$$\frac{\exists y\, Py \qquad \exists y\, Py \to Qa}{Qa}$$

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Finished!

- The conclusion is the sentence at the root.

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- The conclusion is the sentence at the root.
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Conditional proof in informal reasoning.

(1) If it's poison and Quintus took it, then he needs to be readmitted.
(2) It's poison
So (C) if Quintus took it, he need to be readmitted.

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Then (by 2) it's poison *and* he took it.

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Informal proof. Assume Quintus took it.
Then (by 2) it's poison *and* he took it.
Then (by 1 and MP) he needs to be readmitted.

So (by conditional proof) *if* Quintus took it, he needs to be readmitted.

Example

$P, (P \wedge Q) \rightarrow R \vdash Q \rightarrow R$

Example

$P, (P \wedge Q) \to R \vdash Q \to R$ 25

$$\dfrac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix}}{\phi \to \psi} \to\text{Intro}$$

Example

$P, (P \wedge Q) \rightarrow R \vdash Q \rightarrow R$

Assume the first premiss

P

Example

$P, (P \wedge Q) \to R \vdash Q \to R$

$P \qquad Q$

Next assume Q.

- This is the standard way to prove a conditional conclusion.
- We assume the antecedent and prove the consequent.

Example

$P, (P \wedge Q) \rightarrow R \vdash Q \rightarrow R$

Apply $\wedge$Intro.

$$\frac{P \qquad Q}{P \wedge Q}$$

$$\frac{\begin{array}{cc} \vdots & \vdots \\ \phi & \psi \end{array}}{\phi \wedge \psi} \wedge\text{Intro}$$

Example

$P, (P \wedge Q) \rightarrow R \vdash Q \rightarrow R$

Assume the second premiss.

$$\frac{P \qquad Q}{P \wedge Q} \qquad (P \wedge Q) \rightarrow R$$

Example

$P, (P \wedge Q) \to R \vdash Q \to R$

Apply $\to$Elim.

$$\dfrac{\dfrac{P \qquad Q}{P \wedge Q} \qquad (P \wedge Q) \to R}{R}$$

$$\dfrac{\begin{array}{c}\vdots \\ \phi\end{array} \qquad \begin{array}{c}\vdots \\ \phi \to \psi\end{array}}{\psi} \to\text{Elim}$$

Example

$P, (P \wedge Q) \rightarrow R \vdash Q \rightarrow R$

$$\cfrac{\cfrac{P \qquad Q}{P \wedge Q} \qquad (P \wedge Q) \rightarrow R}{R}$$

- Assuming the antecedent Q we've reached the consequent R.
- So we may apply $\rightarrow$Intro

$$\cfrac{\begin{array}{c}[\phi] \\ \vdots \\ \psi\end{array}}{\phi \rightarrow \psi} \rightarrow\text{Intro}$$

Example

$P, (P \wedge Q) \to R \vdash Q \to R$

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- Assuming the antecedent Q we've reached the consequent R.
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$$\begin{array}{c}[\phi] \\ \vdots \\ \cfrac{\psi}{\phi \to \psi}\ \to\text{Intro}\end{array}$$

Example

$P, (P \land Q) \to R \vdash Q \to R$

$$\dfrac{\dfrac{P \qquad [Q]}{P \land Q} \qquad (P \land Q) \to R}{\dfrac{R}{Q \to R}}$$

- Assuming the antecedent Q we've reached the consequent R.
- So we may apply $\to$Intro

$$\dfrac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix}}{\phi \to \psi} \to\text{Intro}$$

- We discharge the assumption of Q.

Example

$P, (P \wedge Q) \to R \vdash Q \to R$

The proof is complete

$$\dfrac{\dfrac{\dfrac{P \qquad [Q]}{P \wedge Q} \qquad (P \wedge Q) \to R}{R}}{Q \to R}$$

Example

$P, (P \wedge Q) \to R \vdash Q \to R$

$$\dfrac{\dfrac{\dfrac{P \qquad [Q]}{P \wedge Q} \qquad (P \wedge Q) \to R}{R}}{Q \to R}$$

The proof is complete

- The conclusion is at the root.

Example

$P, (P \wedge Q) \to R \vdash Q \to R$

$$\dfrac{\dfrac{\dfrac{P \qquad [Q]}{P \wedge Q} \qquad (P \wedge Q) \to R}{R}}{Q \to R}$$

The proof is complete

- The conclusion is at the root.
- The only *undischarged* assumptions are premisses.

Example

$P, (P \wedge Q) \rightarrow R \vdash Q \rightarrow R$

$$\dfrac{\dfrac{\dfrac{P \qquad {\color{red}[Q]}}{P \wedge Q} \qquad (P \wedge Q) \rightarrow R}{R}}{Q \rightarrow R}$$

The proof is complete

- The conclusion is at the root.
- The only *undischarged* assumptions are premisses.
- Discharged assumptions don't need to be amongst the premisses.

We can now define $\Gamma \vdash \phi$.

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Let Γ be a set of sentences and ϕ a sentence.

Definition (Provable)

The sentence ϕ is *provable* from Γ if and only if:

- there is a proof of ϕ with only sentences in Γ as non-discharged assumptions.

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Notation

- $\Gamma \vdash \phi$ is short for ϕ is provable from Γ
- $\vdash \phi$ is short for $\emptyset \vdash \phi$
- $\psi_1, \ldots, \psi_n \vdash \phi$ is short for $\{\psi_1, \ldots, \psi_n\} \vdash \phi$.

Return to the rule of assumption.

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Suppose I assume, the following:

$$\exists x \exists y (Rxy \vee P)$$

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By the rule, this counts as a proof of $\exists x \exists y (Rxy \vee P)$

But it is not an *outright* proof of $\exists x \exists y (Rxy \vee P)$

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- This proof does *not* show $\vdash \exists x \exists y (Rxy \vee P)$

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- Instead it shows $\exists x \exists y(Rxy \vee P) \vdash \exists x \exists y(Rxy \vee P)$

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Rules for $\vee$

The introduction rules are straightforward.

$$\cfrac{\begin{matrix}\vdots\\ \phi\end{matrix}}{\phi \vee \psi}\ \vee\text{Intro1} \qquad\qquad \cfrac{\begin{matrix}\vdots\\ \psi\end{matrix}}{\phi \vee \psi}\ \vee\text{Intro2}$$

The elimination rule is a little more complex.

$$\dfrac{\begin{matrix} \\ \vdots \\ \phi \vee \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ \vdots \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ \vdots \\ \chi \end{matrix}}{\chi} \vee\text{Elim}$$

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Proof by cases in informal reasoning

(1) Either you don't play and you quit or you do something quiet and don't play.
So, (C) you don't play.

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Informal proof. Suppose (1)

Case (i): You don't play and you quit.

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$$\dfrac{\begin{array}{c} \\ \vdots \\ \phi \lor \psi \end{array} \quad \begin{array}{c} [\phi] \\ \vdots \\ \chi \end{array} \quad \begin{array}{c} [\psi] \\ \vdots \\ \chi \end{array}}{\chi} \lor\text{Elim}$$

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Informal proof. Suppose (1)

Case (i): You don't play and you quit. So: you don't play

Case (ii): You do something quiet and don't play.

The elimination rule is a little more complex.

$$\dfrac{\begin{matrix} \\ \vdots \\ \phi \vee \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ \vdots \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ \vdots \\ \chi \end{matrix}}{\chi} \vee\text{Elim}$$

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Case (ii): You do something quiet and don't play. So: you don't play.

The elimination rule is a little more complex.

$$\dfrac{\begin{matrix} \\ \vdots \\ \phi \vee \psi \end{matrix} \quad \begin{matrix} [\phi] \\ \vdots \\ \chi \end{matrix} \quad \begin{matrix} [\psi] \\ \vdots \\ \chi \end{matrix}}{\chi} \vee\text{Elim}$$

Proof by cases in informal reasoning

(1) Either you don't play and you quit or you do something quiet and don't play.
So, (C) you don't play.

Informal proof. Suppose (1)

Case (i): You don't play and you quit. So: you don't play

Case (ii): You do something quiet and don't play. So: you don't play.

Either way then, (C) follows: you don't play.

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$\dfrac{\begin{array}{c} \\ \vdots \\ \phi \vee \psi \end{array} \qquad \begin{array}{c} [\phi] \\ \vdots \\ \chi \end{array} \qquad \begin{array}{c} [\psi] \\ \vdots \\ \chi \end{array}}{\chi} \vee\text{Elim}$$

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P)$$

Assume the premiss

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$\neg P \wedge Q$

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P)$

Case 1: Assume $\neg P \wedge Q$

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \frac{\neg P \wedge Q}{\neg P}$$

Apply $\wedge$Elim1

$$\frac{\begin{matrix}\vdots \\ \phi \wedge \psi\end{matrix}}{\phi}\ \wedge \textit{Elim1}$$

That completes case 1.

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \frac{\neg P \wedge Q}{\neg P} \qquad \exists x\, Qx \wedge \neg P$$

Case 2: Assume $\exists x Qx \wedge \neg P$

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \frac{\neg P \wedge Q}{\neg P} \qquad \frac{\exists x\, Qx \wedge \neg P}{\neg P}$$

Apply $\wedge$Elim1 once more

$$\frac{\begin{array}{c}\vdots\\ \phi \wedge \psi\end{array}}{\phi}\ \wedge\mathit{Elim1}$$

That completes case 2.

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \frac{\neg P \wedge Q}{\neg P} \qquad \frac{\exists x\, Qx \wedge \neg P}{\neg P}$$

We've reached $\neg P$ from each disjunct.
We can now apply $\vee$Elim

$$\frac{\begin{array}{c}\vdots \\ \phi \vee \psi\end{array} \quad \begin{array}{c}[\phi] \\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi] \\ \vdots \\ \chi\end{array}}{\chi}\ \vee\text{Elim}$$

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$\dfrac{(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \dfrac{\neg P \wedge Q}{\neg P} \qquad \dfrac{\exists x\, Qx \wedge \neg P}{\neg P}}{\neg P}$$

We've reached $\neg P$ from each disjunct.
We can now apply $\vee$Elim

$$\dfrac{\begin{matrix} \\ \vdots \\ \phi \vee \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ \vdots \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ \vdots \\ \chi \end{matrix}}{\chi}\ \vee\text{Elim}$$

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$\dfrac{(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \dfrac{[\neg P \wedge Q]}{\neg P} \qquad \dfrac{\exists x\, Qx \wedge \neg P}{\neg P}}{\neg P}$$

We've reached $\neg P$ from each disjunct.
We can now apply $\vee$Elim

$$\dfrac{\begin{matrix} \\ \vdots \\ \phi \vee \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ \vdots \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ \vdots \\ \chi \end{matrix}}{\chi}\ \vee\text{Elim}$$

Example

$(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \vdash \neg P$

$$\dfrac{(\neg P \wedge Q) \vee (\exists x\, Qx \wedge \neg P) \qquad \dfrac{[\neg P \wedge Q]}{\neg P} \qquad \dfrac{[\exists x\, Qx \wedge \neg P]}{\neg P}}{\neg P}$$

We've reached $\neg P$ from each disjunct.
We can now apply $\vee$Elim

$$\dfrac{\begin{matrix} \\ \vdots \\ \phi \vee \psi \end{matrix} \qquad \begin{matrix} [\phi] \\ \vdots \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ \vdots \\ \chi \end{matrix}}{\chi}\ \vee\text{Elim}$$

The rules for ¬

Here are the rules for $\neg$.

$$\dfrac{\begin{matrix}[\phi] & & [\phi] \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\neg\phi}\ \neg\text{Intro}$$

The proof technique is known as *reductio ad absurdum*.

The rules for ¬

Here are the rules for ¬.

$$\dfrac{\begin{matrix}[\phi] & & [\phi] \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\neg\phi}\ \neg\text{Intro} \qquad\qquad \dfrac{\begin{matrix}[\neg\phi] & & [\neg\phi] \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\phi}\ \neg\text{Elim}$$

The proof technique is known as *reductio ad absurdum*.

Example

$\neg(P \to Q) \vdash \neg Q$

Example

$\neg(P \to Q) \vdash \neg Q$

$$\dfrac{\begin{matrix}[\neg\phi] & & [\neg\phi] \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\phi}\,\neg\text{Elim} \qquad \dfrac{\begin{matrix}[\phi] & & [\phi] \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\neg\phi}\,\neg\text{Intro}$$

Example

$\neg(P \to Q) \vdash \neg Q$

$$Q$$

We want to prove $\neg Q$ be *reductio*.
So start by assuming Q, and we'll go for a contradiction.

Example

$\neg(P \to Q) \vdash \neg Q$

Q

$\neg(P \to Q)$

We can always safely assume the premiss.

Example

$\neg(P \to Q) \vdash \neg Q$

$$\dfrac{Q}{P \to Q} \qquad \neg(P \to Q)$$

Next apply →Intro to get a contradiction

$$\dfrac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix}}{\phi \to \psi} \to\text{Intro}$$

(Note this rule can be applied even when we haven't assumed the antecedent)

Example

$\neg(P \to Q) \vdash \neg Q$

$$\dfrac{\dfrac{Q}{P \to Q} \qquad \neg(P \to Q)}{\neg Q}$$

Now we apply ¬Intro

$$\dfrac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\phi] \\ \vdots \\ \neg\psi\end{matrix}}{\neg\phi}\ \neg\text{Intro}$$

And we're done.

Example

$\neg(P \to Q) \vdash \neg Q$

$$\dfrac{\dfrac{[Q]}{P \to Q} \qquad \neg(P \to Q)}{\neg Q}$$

Now we apply ¬Intro

$$\dfrac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\phi] \\ \vdots \\ \neg\psi\end{matrix}}{\neg\phi}\ \neg\text{Intro}$$

And we're done.

Rules for $\leftrightarrow$

These are reminiscent of the rules for $\rightarrow$

$$\dfrac{\begin{matrix}[\phi] & & [\psi] \\ \vdots & & \vdots \\ \psi & & \phi\end{matrix}}{\phi \leftrightarrow \psi} \leftrightarrow\text{Intro}$$

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ \phi \leftrightarrow \psi & & \phi\end{matrix}}{\psi} \leftrightarrow\text{Elim1} \qquad \dfrac{\begin{matrix}\vdots & & \vdots \\ \phi \leftrightarrow \psi & & \psi\end{matrix}}{\phi} \leftrightarrow\text{Elim2}$$

Rules for ∀

$$\dfrac{\begin{array}{c}\vdots \\ \forall v\, \phi\end{array}}{\phi[t/v]}\ \forall\text{Elim}$$

Rules for ∀

$$\dfrac{\begin{matrix}\vdots \\ \forall v\,\phi\end{matrix}}{\phi[t/v]}\ \forall\text{Elim}$$

In this rule:

- ϕ is a formula in which only the variable v occurs freely
- t is a constant
- $\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

Substitution

$\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

- Recall that a free occurrence of v is one not bound by $\exists v$

Compute the following

- $Px[a/x] =$
- $\forall x Px[a/x] =$
- $\forall y(\exists x Px \lor Qx \to Py)[a/x] =$

Substitution

$\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

- Recall that a free occurrence of v is one not bound by $\exists v$

Compute the following

- $Px[a/x] = Pa$
- $\forall x Px[a/x] =$
- $\forall y(\exists x Px \vee Qx \rightarrow Py)[a/x] =$

Substitution

$\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

- Recall that a free occurrence of v is one not bound by $\exists v$

Compute the following

- $Px[a/x] = Pa$
- $\forall x Px[a/x] = \forall x Px$
- $\forall y(\exists x Px \vee Qx \to Py)[a/x] =$

Substitution

$\phi[t/v]$ is the sentence obtained by replacing all free occurrences of v in ϕ by t.

- Recall that a free occurrence of v is one not bound by $\exists v$

Compute the following

- $Px[a/x] = Pa$
- $\forall x Px[a/x] = \forall x Px$
- $\forall y(\exists x Px \vee Qx \to Py)[a/x] = \forall y(\exists x Px \vee Qa \to Py)$

Example

$\forall x\,(Px \to Qx), Pa \vdash Qa$

Example

$\forall x\,(Px \to Qx), Pa \vdash\ Qa$

$$\dfrac{\begin{matrix}\vdots \\ \forall v\,\phi\end{matrix}}{\phi[t/v]}\,\forall\text{Elim}$$

Example

$\forall x\,(Px \to Qx), Pa \vdash Qa$

Assume the first premiss.

$$\forall x\,(Px \to Qx)$$

Example

$\forall x\,(Px \to Qx), Pa \vdash\ Qa$

$$\frac{\forall x\,(Px \to Qx)}{Pa \to Qa}$$

Apply $\forall$Elim

$$\frac{\begin{matrix}\vdots \\ \forall v\,\phi\end{matrix}}{\phi[t/v]}\,\forall\text{Elim}$$

To apply the rule: delete $\forall x$ and by replace all occurrences of x in the formula by the constant a.

Example

$\forall x\,(Px \to Qx), Pa \vdash Qa$

Assume the other premiss

$$Pa \qquad \frac{\forall x\,(Px \to Qx)}{Pa \to Qa}$$

Example

$\forall x\,(Px \to Qx), Pa \vdash Qa$

$$\dfrac{Pa \qquad \dfrac{\forall x\,(Px \to Qx)}{Pa \to Qa}}{Qa}$$

Apply modus ponens.

$$\dfrac{\begin{array}{c}\vdots \\ \phi\end{array} \qquad \begin{array}{c}\vdots \\ \phi \to \psi\end{array}}{\psi} \to\text{Elim}$$

Example

$\forall x\,(Px \to Qx), Pa \vdash Qa$

And we're done

$$\dfrac{Pa \qquad \dfrac{\forall x\,(Px \to Qx)}{Pa \to Qa}}{Qa}$$

Here's the introduction rule for ∀

$$\dfrac{\begin{array}{c}\vdots \\ \phi[t/v]\end{array}}{\forall v\,\phi}\ \forall\text{Intro}$$

side conditions:

(i) the constant t does not occur in ϕ and

(ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Here's the introduction rule for ∀

$$\dfrac{\begin{array}{c}\vdots\\ \phi[t/v]\end{array}}{\forall v\, \phi}\ \forall\text{Intro}$$

side conditions:

(i) the constant t does not occur in ϕ and

(ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every person is either a quorn-lover or a person

Here's the introduction rule for $\forall$

$$\frac{\begin{array}{c}\vdots\\ \phi[t/v]\end{array}}{\forall v\,\phi}\;\forall\text{Intro}$$

side conditions:

(i) the constant t does not occur in ϕ and

(ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every person is either a quorn-lover or a person

Informal proof.

Here's the introduction rule for ∀

$$\dfrac{\begin{matrix}\vdots \\ \phi[t/v]\end{matrix}}{\forall v\, \phi}\ \forall\text{Intro}$$

side conditions:

(i) the constant t does not occur in ϕ and

(ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every person is either a quorn-lover or a person

Informal proof. Let an arbitrary thing be given.
Call it 'Jane Doe'.

Here's the introduction rule for ∀

$$\dfrac{\begin{array}{c}\vdots\\ \phi[t/v]\end{array}}{\forall v\, \phi}\ \forall\text{Intro}$$

side conditions:

(i) the constant t does not occur in ϕ and

(ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every person is either a quorn-lover or a person

Informal proof. Let an arbitrary thing be given.
Call it ‘Jane Doe’.

Clearly, *if* Jane Doe is a person, then Jane Doe is either a quorn-lover or a person.

Here's the introduction rule for $\forall$

$$\dfrac{\begin{array}{c}\vdots \\ \phi[t/v]\end{array}}{\forall v\, \phi}\;\forall\text{Intro}$$

side conditions:

(i) the constant t does not occur in ϕ and

(ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

Informal reasoning with arbitrary names

(C) Every person is either a quorn-lover or a person

Informal proof. Let an arbitrary thing be given.
Call it 'Jane Doe'.

Clearly, *if* Jane Doe is a person, then Jane Doe is either a quorn-lover or a person.

So: every person is either a quorn-lover or a person.

Example

$\vdash \forall z\,(Pz \to Qz \lor Pz)$

Example

$\vdash \forall z\,(Pz \to Qz \vee Pz)$

$$\dfrac{\begin{array}{c}\vdots \\ \phi[t/v]\end{array}}{\forall v\,\phi}\ \forall\text{Intro}$$

Example

$\vdash \forall z\,(Pz \to Qz \vee Pz)$

Pa

I assume Pa.
(We'll try to prove $Pa \to Qa \vee Pa$ without making assumptions about a)

Example

$\vdash \forall z\,(Pz \to Qz \lor Pz)$

$$\frac{Pa}{Qa \lor Pa}$$

Apply $\lor$Intro2.

$$\frac{\begin{array}{c}\vdots\\ \psi\end{array}}{\phi \lor \psi}\ \lor\text{Intro2}$$

Example

$\vdash \forall z\,(Pz \to Qz \vee Pz)$

$$\cfrac{\cfrac{Pa}{Qa \vee Pa}}{Pa \to (Qa \vee Pa)}$$

Apply →Intro

$$\cfrac{\begin{array}{c}[\phi]\\ \vdots \\ \psi\end{array}}{\phi \to \psi}\ \to\text{Intro}$$

Example

$\vdash \forall z\,(Pz \to Qz \lor Pz)$

$$\dfrac{\dfrac{[Pa]}{Qa \lor Pa}}{Pa \to (Qa \lor Pa)}$$

Apply →Intro

$$\dfrac{\begin{array}{c}[\phi] \\ \vdots \\ \psi\end{array}}{\phi \to \psi}\;\to\text{Intro}$$

Example

$\vdash \forall z\,(Pz \to Qz \vee Pz)$

$$\cfrac{\cfrac{\cfrac{[Pa]}{Qa \vee Pa}}{Pa \to (Qa \vee Pa)}}{\forall z\,(Pz \to (Qz \vee Pz))}$$

Finally we want to apply the rule for introducing $\forall$.

$$\begin{array}{c}\vdots\\ \dfrac{\phi[t/v]}{\forall v\,\phi}\ \forall\text{Intro}\end{array}$$

Example

$\vdash \forall z\,(Pz \to Qz \vee Pz)$

$$\dfrac{\dfrac{\dfrac{[Pa]}{Qa \vee Pa}}{Pa \to (Qa \vee Pa)}}{\forall z\,(Pz \to (Qz \vee Pz))}$$

Finally we want to apply the rule for introducing $\forall$.

$$\dfrac{\begin{matrix}\vdots\\ \phi[t/v]\end{matrix}}{\forall v\,\phi}\ \forall\text{Intro}$$

But we also need to check the side conditions are met.
for $t = a$; $\phi = (Pz \to (Qz \vee Pz))$

(i) t does not occur in ϕ

i.e. a does not occur in $(Pz \to (Qz \vee Pz))$

Example

$\vdash \forall z\,(Pz \rightarrow Qz \lor Pz)$

$$\dfrac{\dfrac{\dfrac{[Pa]}{Qa \lor Pa}}{Pa \rightarrow (Qa \lor Pa)}}{\forall z\,(Pz \rightarrow (Qz \lor Pz))}$$

Finally we want to apply the rule for introducing $\forall$.

$$\dfrac{\begin{array}{c}\vdots\\ \phi[t/v]\end{array}}{\forall v\,\phi}\;\forall\text{Intro}$$

But we also need to check the side conditions are met.
for $t = a$; $\phi = (Pz \rightarrow (Qz \lor Pz))$

(i) t does not occur in ϕ

i.e. a does not occur in $(Pz \rightarrow (Qz \lor Pz))$

Example

$\vdash \forall z\,(Pz \to Qz \vee Pz)$

$$\dfrac{\dfrac{\dfrac{\color{red}{[Pa]}}{Qa \vee Pa}}{Pa \to (Qa \vee Pa)}}{\forall z\,(Pz \to (Qz \vee Pz))}$$

Finally we want to apply the rule for introducing $\forall$.

$$\dfrac{\begin{matrix}\vdots \\ \phi[t/v]\end{matrix}}{\forall v\,\phi}\ \forall\text{Intro}$$

But we also need to check the side conditions are met.
for $t = a$; $\phi = (Pz \to (Qz \vee Pz))$

(ii) t does not occur in any undischarged assumption in the proof of $\phi[t/v]$.

i.e. a does not occur in undischarged assumptions.

Rules for $\exists$

The introduction rule is straightforward.

$$\frac{\phi[t/v]}{\exists v\, \phi}\ \exists\text{Intro}$$

Example

$Rcc \vdash \exists y\, Rcy$

Example

$Rcc \vdash \exists y\, Rcy$

Rcc Assume the premiss.

Example

$Rcc \vdash \exists y\, Rcy$

$$Rcc$$

Apply ∃Intro

$$\frac{\phi[t/v]}{\exists v\, \phi}\,\exists\text{Intro}$$

All we need to do is to choose the right ϕ and v

Example

$Rcc \vdash \exists y\, Rcy$

$$\frac{Rcc}{\exists y\, Rcy}$$

Apply ∃Intro

$$\frac{\phi[t/v]}{\exists v\, \phi}\ \exists\text{Intro}$$

All we need to do is to choose the right ϕ and v

Let $\phi = Rcy$, $v = y$

$\phi[c/y] = Rcc$

$\exists v \phi = \exists y Rcy$

The elimination rule is as follows.

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists v\, \phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\, \phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

The elimination rule is as follows.

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists v\, \phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\, \phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

The elimination rule is as follows.

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists v\,\phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof.

The elimination rule is as follows.

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists v\,\phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.

The elimination rule is as follows.

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists v\,\phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.

By (2), Smith is a quindarka.

The elimination rule is as follows.

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\, \phi & \psi \end{array}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\, \phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.

By (2), Smith is a quindarka.
So, something is a quindarka.

The elimination rule is as follows.

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{array}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Dummy names again

(1) Something is an Albanian penny. (2) Every Albanian penny is a quindarka. So, (C) something is a quindarka.

Informal Proof. Let Smith be an Albanian penny.

By (2), Smith is a quindarka.
So, something is a quindarka.

So (C), follows from (1) and (2).

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$Pc$$

The standard way to reason from $\exists x Px$ is to assume Pt (for t a new constant)

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$\forall x\, (Px \to Qx)$$

$$Pc$$

Assume the second premiss.

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$Pc \qquad \frac{\forall x\, (Px \to Qx)}{Pc \to Qc}$$

Apply ∀Elim.

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{Pc \qquad \dfrac{\forall x\, (Px \to Qx)}{Pc \to Qc}}{Qc}$$

Apply $\to$Elim.

Example

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\dfrac{Pc \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}$$

Apply ∃Intro.

Example

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\dfrac{Pc \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}$$

Now we've reached a conclusion assuming Pc (and making no other assumptions about c) we can apply $\exists$Elim.

Example

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\color{red}{\exists x\, Px} \qquad \dfrac{\dfrac{Pc \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}}{\exists x\, Qx}$$

$$\dfrac{\begin{matrix} \\ \vdots \\ \color{red}{\exists v\, \phi} \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

(i) t does not occur in $\exists v\, \phi$

(ii) t does not occur in ψ

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Example

$\exists x\, Px, \forall x\,(Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\exists x\, Px \qquad \dfrac{\dfrac{Pc \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}}{\exists x\, Qx}$$

$$\dfrac{\begin{matrix}\vdots\\ \exists v\,\phi\end{matrix} \qquad \begin{matrix}[\phi[t/v]]\\ \vdots \\ \psi\end{matrix}}{\psi}\ \exists\text{Elim}$$

(i) c does not occur in $\exists x\, Px$

(ii) t does not occur in ψ

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\exists x\, Px \qquad \dfrac{\dfrac{Pc \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}}{\exists x\, Qx}$$

$$\dfrac{\begin{matrix} \vdots \\ \exists v\, \phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

(i) c does not occur in $\exists x\, Px$

(ii) t does not occur in ψ

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\exists x\, Px \qquad \dfrac{\dfrac{Pc \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}}{\color{red}{\exists x\, Qx}}$$

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists v\, \phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\color{red}{\psi}}\ \exists\text{Elim}$$

(i) c does not occur in $\exists x\, Px$

(ii) c does not occur in $\exists x Qx$

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\exists x\, Px \qquad \dfrac{\dfrac{Pc \qquad \dfrac{\forall x\, (Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}}{\exists x\, Qx}$$

$$\dfrac{\begin{array}{c} \vdots \\ \exists v\, \phi \end{array} \qquad \begin{array}{c} [\phi[t/v]] \\ \vdots \\ \psi \end{array}}{\psi}\ \exists\text{Elim}$$

- **(i)** c does not occur in $\exists x\, Px$
- **(ii)** c does not occur in $\exists x Qx$
- **(iii)** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$
\dfrac{\exists x\, Px \qquad \dfrac{\dfrac{{\color{red}Pc} \qquad \dfrac{\forall x\,(Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}}{\exists x\, Qx}
$$

$$
\dfrac{\begin{matrix} \\ \vdots \\ \exists v\, \phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}
$$

(i) c does not occur in $\exists x\, Px$

(ii) c does not occur in $\exists x Qx$

(iii) c does not occur in any undischarged assumption other than Pc in the proof of $\exists x Qx$.

Example

$\exists x\, Px, \forall x\, (Px \to Qx) \vdash \exists x\, Qx$

$$\dfrac{\exists x\, Px \qquad \dfrac{\dfrac{[Pc] \qquad \dfrac{\forall x\, (Px \to Qx)}{Pc \to Qc}}{Qc}}{\exists x\, Qx}}{\exists x\, Qx}$$

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists v\, \phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

(i) c does not occur in $\exists x\, Px$

(ii) c does not occur in $\exists x Qx$

(iii) c does not occur in any undischarged assumption other than Pc in the proof of $\exists x Qx$.

Let Γ be a set of $\mathcal{L}_2$-sentences and ϕ a $\mathcal{L}_2$-sentence.

Two notions of consequence

$\Gamma \vdash \phi$ iff there is a proof of ϕ with only sentences in Γ as non-discharged assumptions.

$\Gamma \vDash \phi$ iff there is no $\mathcal{L}_2$-structure in which all sentences in Γ are true and ϕ is false.

Theorem

(a) Soundness: $\Gamma \vdash \phi$ only if $\Gamma \vDash \phi$

(b) Completeness: $\Gamma \vDash \phi$ only if $\Gamma \vdash \phi$ x

Proof. Elements of Deductive Logic.

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{array}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\, \phi & \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

- ✗ t does not occur in $\exists v\, \phi$
- **(ii)** t does not occur in ψ,
- **(iii)** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$Raa$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\, \phi & \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

- ✗ t does not occur in $\exists v\, \phi$
- **(ii)** t does not occur in ψ,
- **(iii)** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\frac{Raa}{\exists x Rxx}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

✗ t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\exists x\, Rax \qquad \dfrac{Raa}{\exists x Rxx}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} \\ \vdots \\ \exists v\, \phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

- ✗ t does not occur in $\exists v\, \phi$
- **(ii)** t does not occur in ψ,
- **(iii)** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\dfrac{\exists x\, Rax \qquad \dfrac{Raa}{\exists x Rxx}}{\exists x Rxx}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\exists v\,\phi \qquad \begin{matrix}[\phi[t/v]] \\ \vdots \\ \psi\end{matrix}}{\psi}\;\exists\text{Elim}$$

Side conditions:

- **X** t does not occur in $\exists v\,\phi$
- **(ii)** t does not occur in ψ,
- **(iii)** t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\dfrac{\exists x\,Rax \qquad \dfrac{[Raa]}{\exists x Rxx}}{\exists x Rxx}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

✗ t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\dfrac{\exists x\, Rax \qquad \dfrac{[Raa]}{\exists x Rxx}}{\exists x Rxx}$$

But clearly, $\exists x Rax \not\models \exists x Rxx$. Without (i), ND is not sound.

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{array}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{array}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

X t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

✗ t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$Pa$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} \vdots \\ \exists v\,\phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

X t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\exists x\,Px \qquad Pa$$

Why we need the side conditions on ∃Elim

$$\cfrac{\begin{array}{c}\vdots\\ \exists v\,\phi\end{array} \qquad \begin{array}{c}[\phi[t/v]]\\ \vdots\\ \psi\end{array}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

✗ t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\frac{\exists x\, Px \qquad Pa}{Pa}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} \vdots \\ \exists v\,\phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

✗ t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\dfrac{\exists x\,Px \qquad [Pa]}{Pa}$$

Why we need the side conditions on ∃Elim

$$\cfrac{\begin{matrix} \vdots \\ \exists v\, \phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\, \phi$

✗ t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\frac{\exists x\, Px \qquad [Pa]}{Pa}$$

But clearly, $\exists x Px \not\models Pa$. Without (ii), ND is not sound.

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{array}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

(iii) t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

X t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{array}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

✗ t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$Pa$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{array}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

✗ t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$Pa \qquad Qa$$

Why we need the side conditions on ∃Elim

$$\cfrac{\begin{matrix} & & [\phi[t/v]] \\ \vdots & & \vdots \\ \exists v\,\phi & & \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

✗ t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\frac{Pa \qquad Qa}{Pa \wedge Qa}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{array}}{\psi}\;\exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

✗ t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\dfrac{\dfrac{Pa \qquad Qa}{Pa \wedge Qa}}{\exists x(Px \wedge Qx)}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{matrix}}{\psi}\;\exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

✗ t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\exists x\,Px \qquad \dfrac{\dfrac{Pa \qquad Qa}{Pa \wedge Qa}}{\exists x(Px \wedge Qx)}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{array}}{\psi}\;\exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

✗ t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\dfrac{\exists x\,Px \qquad \dfrac{\dfrac{Pa \qquad Qa}{Pa \wedge Qa}}{\exists x(Px \wedge Qx)}}{\exists x(Px \wedge Qx)}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\,\phi & \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

✗ t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\dfrac{\exists x\,Px \qquad \dfrac{\dfrac{[Pa] \qquad Qa}{Pa \wedge Qa}}{\exists x(Px \wedge Qx)}}{\exists x(Px \wedge Qx)}$$

Why we need the side conditions on ∃Elim

$$\dfrac{\begin{matrix} \vdots \\ \exists v\,\phi \end{matrix} \qquad \begin{matrix} [\phi[t/v]] \\ \vdots \\ \psi \end{matrix}}{\psi}\ \exists\text{Elim}$$

Side conditions:

(i) t does not occur in $\exists v\,\phi$

(ii) t does not occur in ψ,

✗ t does not occur in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

$$\dfrac{\exists x\, Px \qquad \dfrac{\dfrac{[Pa] \qquad Qa}{Pa \wedge Qa}}{\exists x(Px \wedge Qx)}}{\exists x(Px \wedge Qx)}$$

But $\exists x Px, Qa \not\models \exists x(Px \wedge Qx)$. Without (iii), ND is not sound.

http://logicmanual.philosophy.ox.ac.uk