INTRODUCTION TO LOGIC

Lecture 2
Syntax and Semantics of Propositional Logic.

Dr. James Studd

Logic is the beginning of wisdom.
Thomas Aquinas

Outline

1 Syntax vs Semantics.
2 Syntax of $L_1$.
3 Semantics of $L_1$.
4 Truth-table methods.

Syntax

Syntax is all about expressions: words and sentences.

Examples of syntactic claims
- ‘Bertrand Russell’ is a proper noun.
- ‘likes logic’ is a verb phrase.
- ‘Bertrand Russell likes logic’ is a sentence.
- Combining a proper noun and a verb phrase in this way makes a sentence.

Semantics

Semantics is all about meanings of expressions.

Examples of semantic claims
- ‘Bertrand Russell’ refers to a British philosopher.
- ‘Bertrand Russell’ refers to Bertrand Russell.
- ‘likes logic’ expresses a property Russell has.
- ‘Bertrand Russell likes logic’ is true.
Use vs Mention

Note our use of quotes to talk about expressions.

‘Bertrand Russell’ refers to Bertrand Russell.

Mention

- The first occurrence of ‘Bertrand Russell’ is an example of mention.
- This occurrence (with quotes) mentions—refers to—an expression.

Use

- The second occurrence of ‘Bertrand Russell’ is an example of use.
- This occurrence (without quotes) uses the expression to refer to a man.

Syntax: English vs. $\mathcal{L}_1$.

English has many different sorts of expression.

Some expressions of English

2. Connectives: ‘it is not the case that’, ‘and’, etc..
4. Verb phrases: ‘likes logic’, ‘like conceptual analysis’, etc..
5. Also: nouns, verbs, pronouns, etc., etc., etc..

$\mathcal{L}_1$ has just two sorts of basic expression.

Some basic expressions of $\mathcal{L}_1$

1. Sentence letters: e.g. ‘P’, ‘Q’.
2. Connectives: e.g. ‘$\neg$’, ‘$\land$’.

Combining sentences and connectives makes new sentences.

Some complex sentences

- ‘It is not the case that’ and ‘Bertrand Russell likes logic’ make: ‘It is not the case that Bertrand Russell likes logic’.
- ‘$\neg$’ and ‘P’ make: ‘$\neg P$’.
- ‘Bertrand Russell likes logic’ and ‘and’ and ‘Philosophers like conceptual analysis’ make: ‘Bertrand Russell likes logic and philosophers like conceptual analysis’.
- ‘P’, ‘$\land$’ and ‘Q’ make: ‘(P $\land$ Q)’.

Logic convention: no quotes around $\mathcal{L}_1$-expressions.

- P, $\land$ and Q make: (P $\land$ Q).

Connectives

Here’s the full list of $\mathcal{L}_1$-connectives.

<table>
<thead>
<tr>
<th>name</th>
<th>in English</th>
<th>symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>conjunction</td>
<td>and</td>
<td>$\land$</td>
</tr>
<tr>
<td>disjunction</td>
<td>or</td>
<td>$\lor$</td>
</tr>
<tr>
<td>negation</td>
<td>it is not the case that</td>
<td>$\neg$</td>
</tr>
<tr>
<td>arrow</td>
<td>if ... then</td>
<td>$\rightarrow$</td>
</tr>
<tr>
<td>double arrow</td>
<td>if and only if</td>
<td>$\leftrightarrow$</td>
</tr>
</tbody>
</table>
The syntax of $L_1$

Here’s the official definition of $L_1$-sentence.

**Definition**

(i) All sentence letters are sentences of $L_1$:
   - $P, Q, R, P_1, Q_1, R_1, P_2, Q_2, R_2, P_3, \ldots$

(ii) If $\phi$ and $\psi$ are sentences of $L_1$, then so are:
   - $\neg \phi$
   - $(\phi \land \psi)$
   - $(\phi \lor \psi)$
   - $(\phi \rightarrow \psi)$
   - $(\phi \leftrightarrow \psi)$

(iii) Nothing else is a sentence of $L_1$.

**Greek letters:** $\phi$ (‘PHI’) and $\psi$ (‘PSI’): not part of $L_1$.

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How to build a sentence of $L_1$

**Example**

The following is a sentence of $L_1$:

$$\neg \neg (((P \land Q) \rightarrow (P \lor \neg R_{45})) \leftrightarrow \neg ((P_3 \lor R) \lor R))$$

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Object vs. Metalanguage

I mentioned that $\phi$ and $\psi$ are not part of $L_1$.

- $\neg P$ is a $L_1$-sentence.
- $\neg \phi$ describes many $L_1$-sentences (but is not one itself).
  - e.g. $\neg P$, $\neg (Q \lor R)$, $\neg (P \leftrightarrow (Q \lor R))$...

$\phi$ and $\psi$ are part of the metalanguage, not the object one.

**Object language**

The object language is the one we’re theorising about.

- The object language is $L_1$.

**Metalanguage**

The metalanguage is the one we’re theorising in.

- The metalanguage is (augmented) English.

$\phi$ and $\psi$ are used as variables in the metalanguage, in order to generalise about sentences of the object language.

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Bracketing conventions

There are conventions for dropping brackets in $L_1$.

Some are similar to rules used for $+$ and $\times$ in arithmetic.

**Example in arithmetic**

- $4 + 5 \times 3$ does not abbreviate $(4 + 5) \times 3$.
- $\times$ ‘binds more strongly’ than $+$.
  - $4 + 5 \times 3$ abbreviates $4 + (5 \times 3)$.

**Examples in $L_1$**

- $\land$ and $\lor$ bind more strongly than $\rightarrow$ and $\leftrightarrow$.
  - $(P \rightarrow Q \land R)$ abbreviates $(P \rightarrow (Q \land R))$.
- One may drop outer brackets.
  - $P \land (Q \rightarrow \neg P_4)$ abbreviates $(P \land (Q \rightarrow \neg P_4))$.
- One may drop brackets on strings of $\land$ or $\lor$.
  - $(P \land Q \land R)$ abbreviates $((P \land Q) \land R)$.
Semantics

Recall the characterisation of validity from week 1.

Characterisation
An argument is **logically valid** if and only if there is no interpretation of subject-specific expressions under which:

(i) the premisses are all true, and
(ii) the conclusion is false.

We’ll adapt this characterisation to $L_1$.

- Logical expressions: $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.
- Subject specific expressions: $P, Q, R, \ldots$
- Interpretation: $L_1$-structure.

$L_1$-structures

We interpret sentence letters by assigning them truth-values: either $T$ for True or $F$ for False.

**Definition**
An $L_1$-structure is an assignment of exactly one truth-value ($T$ or $F$) to every sentence letter of $L_1$.

**Examples**
One may think of an $L_1$-structure as an infinite list that provides a value $T$ or $F$ for every sentence letter.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$P_1$</th>
<th>$Q_1$</th>
<th>$R_1$</th>
<th>$P_2$</th>
<th>$Q_2$</th>
<th>$R_2$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ :</td>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$B$ :</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

We use $A, B, \text{etc.}$ to stand for $L_1$-structures.

Truth-values of complex sentences 1/3

$L_1$-structures **only** directly specify truth-values for $P, Q, R, \ldots$

- The logical connectives have fixed meanings.
- These determine the truth-values of complex sentences.
- Notation: $|\phi |_A$ is the truth-value of $\phi$ under $A$.

**Truth-conditions for $\neg$**
The meaning of $\neg$ is summarised in its **truth table**.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\neg \phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

In words: $|\neg \phi |_A = T$ if and only if $|\phi |_A = F$.

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\neg \phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

Worked example 1

$|\phi |_A$ is the truth-value of $\phi$ under $A$.

Compute the following truth-values.

Let the structure $A$ be partially specified as follows.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$P_1$</th>
<th>$Q_1$</th>
<th>$R_1$</th>
<th>$P_2$</th>
<th>$Q_2$</th>
<th>$R_2$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
<td>$F$</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

Compute:

$|P |_A$ = $|Q |_A$ = $|R_1 |_A$ =
$|\neg P |_A$ = $|\neg Q |_A$ = $|\neg R_1 |_A$ =
$|\neg \neg P |_A$ = $|\neg \neg Q |_A$ = $|\neg \neg R_1 |_A$ =
Truth-values of complex sentences 2/3

Truth-conditions for $\land$ and $\lor$

The meanings of $\land$ and $\lor$ are given by the truth tables:

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$(\phi \land \psi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$(\phi \lor \psi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

$|(\phi \land \psi)|_A = T$ if and only if $|\phi|_A = T$ and $|\psi|_A = T$.

$|(\phi \lor \psi)|_A = T$ if and only if $|\phi|_A = T$ or $|\psi|_A = T$ (or both).

Truth-values of complex sentences 3/3

Truth-conditions for $\rightarrow$ and $\leftrightarrow$

The meanings of $\rightarrow$ and $\leftrightarrow$ are given by the truth tables:

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$(\phi \rightarrow \psi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$(\phi \leftrightarrow \psi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

$|(\phi \rightarrow \psi)|_A = T$ if and only if $|\phi|_A = F$ or $|\psi|_A = T$.

$|(\phi \leftrightarrow \psi)|_A = T$ if and only if $|\phi|_A = |\psi|_A$.

Worked example 2

Let $|P|_B = T$ and $|Q|_B = F$.

Compute $|\neg(P \rightarrow Q) \rightarrow (P \land Q)|_B$

What is the truth value of $\neg(P \rightarrow Q) \rightarrow (P \land Q)$ under $B$?

1. $|(P \rightarrow Q)|_B = F$ and $|(P \land Q)|_B = F$
2. $|\neg(P \rightarrow Q)|_B = T$
3. $|\neg(P \rightarrow Q) \rightarrow (P \land Q)|_B = F$

For actual calculations it’s usually better to use tables.

Suppose $|P|_B = T$ and $|Q|_B = F$.

Compute $|\neg(P \rightarrow Q) \rightarrow (P \land Q)|_B$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$\neg(P \rightarrow Q) \rightarrow (P \land Q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$\psi$</th>
<th>$(\phi \land \psi)$</th>
<th>$(\phi \rightarrow \psi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
Using the same technique we can fill out the full truth table for \( \neg(P \rightarrow Q) \rightarrow (P \land Q) \):

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
  P & Q & \neg(P \rightarrow Q) & (P \land Q) \\
  \hline
  T & T & F & T & T & T & T & T & T & T \\
  T & F & F & T & F & T & F & T & F & F \\
  F & T & T & F & T & T & F & T & F & F \\
  F & F & T & F & T & F & F & T & F & F \\
\end{array}
\]

The main column (underlined) gives the truth-value of the whole sentence.

**Validity**

Let \( \Gamma \) be a set of sentences of \( \mathcal{L}_1 \) and \( \phi \) a sentence of \( \mathcal{L}_1 \).

**Definition**

The argument with all sentences in \( \Gamma \) as premisses and \( \phi \) as conclusion is valid if and only if there is no \( \mathcal{L}_1 \)-structure under which:

(i) all sentences in \( \Gamma \) are true; and

(ii) \( \phi \) is false.

Notation: when this argument is valid we write \( \Gamma \models \phi \).

\{P \rightarrow \neg Q, Q\} \models \neg P \text{ means that the argument whose premises are } P \rightarrow \neg Q \text{ and } Q, \text{ and whose conclusion is } \neg P \text{ is valid.}

Also written: \( P \rightarrow \neg Q, Q \models \neg P \)

**Worked example 3**

We can use truth-tables to show that \( \mathcal{L}_1 \)-arguments are valid.

**Example**

Show that \( \{P \rightarrow \neg Q, Q\} \models \neg P \).

\[
\begin{array}{c|c|c|c|c}
  P & Q & P \rightarrow \neg Q & \neg P \\
  \hline
  T & T & T & F \\
  T & F & F & T \\
  F & T & T & T \\
  F & F & T & T \\
\end{array}
\]

Rows correspond to interpretations.

One needs to check that there is no row in which all the premisses are assigned T and the conclusion is assigned F.

**Other logical notions**

**Definition**

A sentence \( \phi \) of \( \mathcal{L}_1 \) is **logically true** (a tautology) iff:

- \( \phi \) is true under all \( \mathcal{L}_1 \)-structures.

e.g. \( P \lor \neg P \), and \( P \rightarrow P \) are tautologies.

**Truth tables of tautologies**

Every row in the main column is a T.

\[
\begin{array}{c|c|c}
  P & P \lor \neg P & P \rightarrow P \\
  \hline
  T & T & T \\
  T & F & T \\
  F & T & T \\
\end{array}
\]
Definition
A sentence \( \phi \) of \( \mathcal{L}_1 \) is a contradiction iff:
- \( \phi \) is not true under any \( \mathcal{L}_1 \)-structure.

e.g. \( P \land \neg P \), and \( \neg (P \rightarrow P) \) are contradictions.

Truth tables of contradictions
Every row in the main column is an F.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( P \land \neg P )</th>
<th>( \neg (P \rightarrow P) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Definition
Sentences \( \phi \) and \( \psi \) are logically equivalent iff:
- \( \phi \) and \( \psi \) are true in exactly the same \( \mathcal{L}_1 \)-structures.

- \( P \) and \( \neg \neg P \) are logically equivalent.
- \( P \land Q \) and \( \neg (\neg P \lor \neg Q) \) are logically equivalent.

Truth tables of logical equivalents
The truth-values in the main columns agree.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \land Q )</th>
<th>( \neg (\neg P \lor \neg Q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Worked example 4

Example
Show that the sentence \( (P \rightarrow (\neg Q \land R)) \lor P \) is a tautology.

Method 1: Full truth table
- Write out the truth table for \( (P \rightarrow (\neg Q \land R)) \lor P \).
- Check there’s a T in the every row of the main column.

Method 2: Backwards truth table.
- Put an F in the main column.
- Work backwards to show this leads to a contradiction.
Worked example 5

**Example**
Show that \( P \leftrightarrow \neg Q \models \neg (P \leftrightarrow Q) \)

**Method 1: Full truth table**
- Write out the full truth table.
- Check there’s no row in which the main column of the premiss is T and the main column of the conclusion is F.

**Method 2: Backwards truth table**
- Put a T in the main column of the premiss and an F in the main column of the conclusion.
- Work backwards to obtain a contradiction.

\[
\begin{array}{c|c|c|c}
P & Q & P \leftrightarrow \neg Q & \neg (P \leftrightarrow Q) \\
\hline
\phi & \neg \phi & \phi & \psi & (\phi \leftrightarrow \psi) \\
T & F & T & T & T \\
F & T & F & F & T \\
F & F & F & T & T \\
\end{array}
\]