Definitions from *The Logic Manual*

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1 Sets, Relations and Arguments

Binary relation: A set is a binary relation iff it contains only ordered pairs.

Types of binary relation: A binary relation R is
- (i) *reflective* on a set S iff for all elements d of S the pair $\langle d, d\rangle$ is an element of R;
- (ii) *symmetric* on a set S iff for all elements d, e of S: if $\langle d, e\rangle \in R$ then $\langle e, d\rangle \in R$;
- (iii) *asymmetric* on a set S iff for no elements d, e of S: $\langle d, e\rangle \in R$ and $\langle e, d\rangle \in R$;
- (iv) *antisymmetric* on a set S iff for no two distinct elements d, e of S: $\langle d, e\rangle \in R$ and $\langle e, d\rangle \in R$;
- (v) *transitive* on a set S iff for all elements d, e, f of S: if $\langle d, e\rangle \in R$ and $\langle e, f\rangle \in R$, then $\langle d, f\rangle \in R$.

Binary relations simpliciter: A binary relation R is
- (i) *symmetric* iff it is symmetric on all sets;
- (ii) *assymmetric* iff it is asymmetric on all sets;
- (iii) *antisymmetric* iff it is antisymmetric on all sets;
- (iv) *transitive* iff it is transitive on all sets.

Equivalence relation: A binary relation R is an equivalence relation on S iff R is reflexive on S, symmetric on S and transitive on S.

Function: A binary relation R is a function iff for all d, e, f: if $\langle d, e\rangle \in R$ and $\langle d, f\rangle \in R$ then $e = f$.

Domain, range, into:
- (i) The *domain* of a function R is the set $\{d$: there is an e such that $\langle d, e\rangle \in R\}$.
- (ii) The *range* of a function R is the set $\{e$: there is a d such that $\langle d, e\rangle \in R\}$.
- (iii) R is a function into the set M iff all elements of the range of the function are in M.

Function notation: If d is in the domain of a function R one writes $R(d)$ for the unique object e such that $\langle d, e\rangle$ is in R.

***n*-ary relation**: An n-place relation is a set containing only n-tuples. An n-place relation is called a relation of arity n.

Argument: An argument consists of a set of declarative sentences (the premises) and a declarative sentence (the conclusion) marked as the concluded sentence.

Logical validity: An argument is logically valid iff there is no interpretation under which the premises are all true and the conclusion false.

Consistency: A set of sentences is logically consistent iff there is at least one interpretation under which all sentences of the set are true.

Logical truth: A setence is logically true iff it is true under any interpretation.

Contradiction: A sentence is a contradiction iff it is false under all interpretations.

Logical equivalence: Sentences are logically equivalent iff they are true under exactly the same interpretations.

2 Syntax and Semantics of Propositional Logic

Sentence letters: $P,\ Q,\ R,\ P_1,\ Q_1,\ R_1,\ P_2,\ Q_2,\ R_2$ and so on are sentence letters.

Sentence of $\mathcal{L}_1$:
(i) All sentence letters are sentences of $\mathcal{L}_1$.
(ii) If ϕ and ψ are sentences of $\mathcal{L}_1$, then $\neg\phi,\ (\phi\wedge\psi),\ (\phi\vee\psi),\ (\phi\rightarrow\psi)$ and $(\phi\leftrightarrow\psi)$ are sentences of $\mathcal{L}_1$.
(iii) Nothing else is a sentence of $\mathcal{L}_1$.

Bracketing Convention:
1 The outer brackets may be omitted from a sentence that is not part of another sentence.
2 The inner set of brackets may be omitted from a sentence of the form $((\phi\wedge\psi)\wedge\chi)$ and analgously for $\vee$.
3 Suppose $\diamond\in\{\wedge,\ \vee\}$ and $\circ\in\{\rightarrow,\ \leftrightarrow\}$. Then if $(\phi\circ(\psi\diamond\chi))$ or $((\phi\diamond\psi)\circ\chi)$ occurs as part of the sentence that is to be abbreviated, the inner set of brackets may be omitted.

$\mathcal{L}_1$-structure: An $\mathcal{L}_1$-structure is an assignment of exactly one truth-value (T or F) to every sentence letter of $\mathcal{L}_1$.

Truth in an $\mathcal{L}_1$-structure: Let $\mathcal{A}$ be some $\mathcal{L}_1$-structure. Then $|\ldots|_{\mathcal{A}}$ assigns either T or F to every sentence of $\mathcal{L}_1$ in the following way.
(i) If ϕ is a sentence letter, $|\phi|_{\mathcal{A}}$ is the truth-value assigned to ϕ by the $\mathcal{L}_1$-structure $\mathcal{A}$
(ii) $|\neg\phi|_{\mathcal{A}}=T$ iff $|\phi|_{\mathcal{A}}=F$
(iii) $|\phi\wedge\psi|_{\mathcal{A}}=T$ iff $|\phi|_{\mathcal{A}}=T$ and $|\psi|_{\mathcal{A}}=T$
(iv) $|\phi\vee\psi|_{\mathcal{A}}=T$ iff $|\phi|_{\mathcal{A}}=T$ or $|\psi|_{\mathcal{A}}=T$
(v) $|\phi\rightarrow\psi|_{\mathcal{A}}=T$ iff $|\phi|_{\mathcal{A}}=F$ or $|\psi|_{\mathcal{A}}=T$
(vi) $|\phi\leftrightarrow\psi|_{\mathcal{A}}=T$ iff $|\phi|_{\mathcal{A}}=|\psi|_{\mathcal{A}}$

Truth tables:

ϕ	$\neg\phi$
T	F
F	T

ϕ	ψ	$(\phi\wedge\psi)$	$\phi\vee\psi$	$(\phi\rightarrow\psi)$	$(\phi\leftrightarrow\psi)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Logical truth etc. ($\mathcal{L}_1$ version):
(i) A sentence ϕ of $\mathcal{L}_1$ is logically true iff ϕ is true in all $\mathcal{L}_1$-structures.
(ii) A sentence ϕ of $\mathcal{L}_1$ is a contradiction iff ϕ is not true in any $\mathcal{L}_1$-structures.
(iii) A sentence ϕ and a sentence ψ of $\mathcal{L}_1$ are logically equivalent iff ϕ and ψ are true in exactly the same $\mathcal{L}_1$-structures.

Validity ($\mathcal{L}_1$ version): Let Γ be a set of sentences of $\mathcal{L}_1$ and ϕ a sentence of $\mathcal{L}_1$. The argument with all sentences in Γ as premisses and ϕ as conclusion is valid iff there is no $\mathcal{L}_1$-structure in which all sentences in Γ are true and ϕ is false.

Counterexamples: An $\mathcal{L}_1$-structure is a counterexample to the argument with Γ as the set of premisses and ϕ as the conclusion iff for all $\gamma \in \Gamma$ we have $|\gamma|_{\mathcal{A}} = T$ but $|\phi|_{\mathcal{A}} = F$.

Semantic Consistency: A set Γ of $\mathcal{L}_1$-sentences is semantically consistent iff there is an $\mathcal{L}_1$-structure $\mathcal{A}$ such that for all sentence $\gamma \in \Gamma$ we have $|\gamma|_{\mathcal{A}} = T$. A set Γ of $\mathcal{L}_1$-sentences is semantically inconsistent iff Γ is not semantically consistent.

3 Formalization in Propositional Logic

Truth-functionality: A connective is truth-functional iff the truth-value of the compound sentence cannot be changed by replacing a direct subsentence with another sentence having the same truth-value.

Scope of a connective in $\mathcal{L}_1$: The scope of an occurrence of a connective in a sentence ϕ of $\mathcal{L}_1$ is the occurrence of the smallest subsentence of ϕ that contains this occurrence of the connective.

Logical truth etc. (propositional version):
(i) An English sentence is a tautology iff its formalization in propositional logic is logically true.
(ii) An English sentence is a contradiction iff its formalization in propositional logic is a contradiction.
(iii) An set of English sentences is propositionally consistent iff the set of all their formalizations in propositional logic is semantically consistent.

Propositional validity: An argument in English is propositionally valid iff its formalization in $\mathcal{L}_1$ is valid.

4 The Syntax of Predicate Logic

Predicate letters: All expressions of the form P_n^k, Q_n^k, R_n^k are predicate letters where k and n are either missing or a numeral '1', '2' . . . .

Arity: The value of the upper index of a predicate letter is called its arity. If a predicate letter does not have an upper index its arity is 0.

Constants: $a,\ b,\ c,\ a_1,\ b_1,\ c_1,\ a_2,\ b_2,\ c_2,\ \ldots$ are constants.

Variables: $x,\ y,\ z,\ x_1,\ y_1,\ z_1,\ x_2,\ y_2,\ z_2,\ \ldots$ are variables.

Atomic formulae of $\mathcal{L}_2$: If Z is a predicate letter of arity n and each of $t_1,\ \ldots,\ t_n$ is a variable or constant, then $Zt_1 \ldots t_n$ is an atomic formula of $\mathcal{L}_2$.

Quantifier: A quantifier is an expression $\forall v$ or $\exists v$ where v is a variable.

Formulae of $\mathcal{L}_2$:
(i) All atomic formulae of $\mathcal{L}_2$ are formulae of $\mathcal{L}_2$.
(ii) If ϕ and ψ are formulae of $\mathcal{L}_2$ then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$ and $(\phi \leftrightarrow \psi)$ are formulae of $\mathcal{L}_2$.

(iii) If v is a variable and ϕ is a formula then $\forall v\phi$ and $\exists v\phi$ are formulae of $\mathcal{L}_2$.
(iv) Nothing else is a formula of $\mathcal{L}_2$.

Free occurrence of a variable:
(i) All occurrences of variables in atomic formulae are free.
(ii) The occurences of a varaiable that are free in ϕ and ψ are also free in $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, and $\phi \leftrightarrow \psi$.
(iii) In a formula $\forall v\phi$ or $\exists v\phi$ no occurrence of the variable v is free; all occurrences of variables other than v that are free in ϕ are also free in $\forall v\phi$ and $\exists v\phi$.

An occurrence of a variable is bound in a formula iff it is not free.
A variable occurs freely in a formula iff there is at least one free occurrence of the variable in the formula.

Sentence of $\mathcal{L}_2$: A formula of $\mathcal{L}_2$ is a sentence of $\mathcal{L}_2$ iff no variable occurs freely in the formula.

5 The Semantics of Predicate Logic

$\mathcal{L}_2$-structure: An $\mathcal{L}_2$-structure is an ordered pair $\langle D, I\rangle$ where D is some non-empty set and I is a function from the set of all constants, sentence letters, and predicate letters such that
- the value of every constant is an element of D
- the value of every sentence letter is a truth-value T or F
- the value of every n-ary predicate letter is an n-ary relation.

Variable assignment: A variable assignment over an $\mathcal{L}_2$-structure $\mathcal{A}$ assigns an element of the domain $D_{\mathcal{A}}$ of $\mathcal{A}$ to each variable.

Satisfaction: Assume $\mathcal{A}$ is an $\mathcal{L}_2$-structure, α is a variable assignment over $\mathcal{A}$, ϕ and ψ are formulae of $\mathcal{L}_2$, and v is a variable. For a sentence letter ϕ either $|\phi|^{\alpha}_{\mathcal{A}} = T$ or $|\phi|^{\alpha}_{\mathcal{A}} = F$ obtains. Formulae other than sentence letters receive the following semantic values.
(i) $|\Phi t_1 \ldots t_n|^{\alpha}_{\mathcal{A}} = T$ iff $\langle |t_1|^{\alpha}_{\mathcal{A}}, \ldots , |t_n|^{\alpha}_{\mathcal{A}}\rangle \in |\Phi|^{\alpha}_{\mathcal{A}}$, where Φ is an n-ary predicate letter for $n \geqslant 1$ and each of $t_1, \ldots, t_n$ is either a variable or a constant
(ii) $|\neg\phi|^{\alpha}_{\mathcal{A}} = T$ iff $|\phi|^{\alpha}_{\mathcal{A}} = F$
(iii) $|\phi \wedge \psi|^{\alpha}_{\mathcal{A}} = T$ iff $|\phi|^{\alpha}_{\mathcal{A}} = T$ and $|\psi|^{\alpha}_{\mathcal{A}} = T$
(iv) $|\phi \vee \psi|^{\alpha}_{\mathcal{A}} = T$ iff $|\phi|^{\alpha}_{\mathcal{A}} = T$ or $|\psi|^{\alpha}_{\mathcal{A}} = T$
(v) $|\phi \to \psi|^{\alpha}_{\mathcal{A}} = T$ iff $|\phi|^{\alpha}_{\mathcal{A}} = F$ or $|\psi|^{\alpha}_{\mathcal{A}} = T$
(vi) $|\phi \leftrightarrow \psi|^{\alpha}_{\mathcal{A}} = T$ iff $|\phi|^{\alpha}_{\mathcal{A}} = |\psi|^{\alpha}_{\mathcal{A}}$
(vii) $|\forall v\phi|^{\alpha}_{\mathcal{A}} = T$ iff $|\phi|^{\beta}_{\mathcal{A}} = T$ for all variable assignments β over $\mathcal{A}$ differeing from α in v at most
(viii) $|\exists v\phi|^{\alpha}_{\mathcal{A}} = T$ iff $|\phi|^{\beta}_{\mathcal{A}} = T$ for at least one variable assignment β over $\mathcal{A}$ differeing from α in v at most

Truth: A sentence ϕ is true in an $\mathcal{L}_2$-structure $\mathcal{A}$ iff $|\phi|^{\alpha}_{\mathcal{A}} = T$ for all variable assignments α over $\mathcal{A}$.

Logical truth etc. ($\mathcal{L}_2$ version)

(i) A sentence ϕ of $\mathcal{L}_2$ is logically true iff ϕ is true in all $\mathcal{L}_2$-structures.
(ii) A sentence ϕ of $\mathcal{L}_2$ is a contradiction iff ϕ is not true in any $\mathcal{L}_2$-structures.
(iii) Sentences ϕ and ψ of $\mathcal{L}_2$ are logically equivalent iff both are true in exactly the same $\mathcal{L}_2$-structures.
(iv) A set Γ of $\mathcal{L}_2$-setences is semantically consistent iff there is an $\mathcal{L}_2$-structure $\mathcal{A}$ in which all sentences in Γ are true. A set of $\mathcal{L}_2$-sentences is semantically inconsistent iff it is not semantically consistent.

Validity ($\mathcal{L}_2$version): Let Γ be a set of sentences of $\mathcal{L}_2$ and ϕ a sentence of $\mathcal{L}_2$. The argument with all sentences in Γ as premisses and ϕ as conclusion is valid iff there is no $\mathcal{L}_2$ structure in which all sentences in Γ are true and ϕ is false. This is abbreviated as $\Gamma \models \phi$.

6 Natural Deduction

Propositional Logic Rules

$$\frac{\begin{matrix}\vdots & \vdots\\ \phi & \psi\end{matrix}}{\phi \land \psi}\ \land\text{Intro}$$

$$\frac{\begin{matrix}\vdots\\ \phi \land \psi\end{matrix}}{\phi}\ \land\text{Elim}_1 \qquad \frac{\begin{matrix}\vdots\\ \phi \land \psi\end{matrix}}{\psi}\ \land\text{Elim}_2$$

$$\frac{\begin{matrix}\vdots\\ \phi\end{matrix}}{\phi \lor \psi}\ \lor\text{Intro}_1 \qquad \frac{\begin{matrix}\vdots\\ \psi\end{matrix}}{\phi \lor \psi}\ \lor\text{Intro}_2$$

$$\frac{\begin{matrix} & [\phi] & [\psi]\\ \vdots & \vdots & \vdots\\ \phi \lor \psi & \chi & \chi\end{matrix}}{\chi}\ \lor\text{Elim}$$

$$\frac{\begin{matrix}[\phi]\\ \vdots\\ \psi\end{matrix}}{\phi \to \psi}\ \to\text{Intro} \qquad \frac{\begin{matrix}\vdots & \vdots\\ \phi & \phi \to \psi\end{matrix}}{\psi}\ \to\text{Elim}$$

$$\frac{\begin{matrix}[\phi] & [\phi]\\ \vdots & \vdots\\ \psi & \neg\psi\end{matrix}}{\neg\phi}\ \neg\text{Intro} \qquad \frac{\begin{matrix}[\neg\phi] & [\neg\phi]\\ \vdots & \vdots\\ \psi & \neg\psi\end{matrix}}{\phi}\ \neg\text{Elim}$$

$$\dfrac{\begin{array}{cc}[\phi] & [\psi] \\ \vdots & \vdots \\ \psi & \phi\end{array}}{\phi \leftrightarrow \psi}\ \leftrightarrow\text{Intro}$$

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \phi \leftrightarrow \psi & \phi\end{array}}{\psi}\ \leftrightarrow\text{Elim}_1 \qquad \dfrac{\begin{array}{cc}\vdots & \vdots \\ \phi \leftrightarrow \psi & \psi\end{array}}{\phi}\ \leftrightarrow\text{Elim}_2$$

Predicate Logic Rules

$$\dfrac{\begin{array}{c}\vdots \\ \phi[t/v]\end{array}}{\forall v\phi}\ \forall\text{Intro}$$

provided that the constant t does not occur in ϕ or in any undischarged assumption in the proof of $\phi[t/v]$.

$$\dfrac{\begin{array}{c}\vdots \\ \forall v\phi\end{array}}{\phi[t/v]}\ \forall\text{Elim} \qquad \dfrac{\begin{array}{c}\vdots \\ \phi[t/v]\end{array}}{\exists v\phi}\ \exists\text{Intro}$$

$$\dfrac{\begin{array}{cc} & [\phi[t/v]] \\ \vdots & \vdots \\ \exists v\phi & \psi\end{array}}{\psi}\ \exists\text{Elim}$$

provided that the constant t does not occur in $\exists v\phi$ or in ψ or in any undischarged assumption other than $\phi[t/v]$ in the proof of ψ.

Identity Rules

$$\dfrac{[t = t]}{\vdots}\ =\!\text{Intro}$$

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \phi[s/v] & s = t\end{array}}{\phi[t/v]}\ =\!\text{Elim} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots \\ \phi[s/v] & t = s\end{array}}{\phi[t/v]}\ =\!\text{Elim}$$

7 Formalization in Predicate Logic

Syntactic consistency: A set Γ of $\mathcal{L}_2$-sentences is syntactically consistent iff there is a sentence ϕ such that $\Gamma \nvdash \phi$.

Scope of a quantifier or connective in $\mathcal{L}_2$: The scope of an occurrence of a quantifiers or a connective in a setence ϕ of $\mathcal{L}_2$ is the occurrence of the smallest $\mathcal{L}_2$-formula that contains that occurrence of the quantifier or connective and is part of ϕ.

Logical truth etc. (predicate version):

(i) An English sentence is logically true in predicate logic iff its formalization in predicate logic is logically true.

(ii) An English sentence is a contradiction in predicate logic iff its formalization in predicate logic is a contradiction.

(iii) A set of English sentences is consistent in predicate logic iff the set of their formalizations in predicate logic is semantically consistent.

Validity (predicate version): An argument in English is valid in predicate logic iff its formalization in the language $\mathcal{L}_2$ of predicate logic is valid.

8 Identity and Definite Descriptions

Atomic formulae of $\mathcal{L}_=$: All atomic formulae of $\mathcal{L}_2$ are atomic formulae of $\mathcal{L}_=$. Furthermore, if s and t are variables or constants then $s = t$ is an atomic formla of $\mathcal{L}_=$.

Formulae of $\mathcal{L}_=$:

(i) All atomic formulae of $\mathcal{L}_=$ are formulae of $\mathcal{L}_=$.

(ii) If ϕ and ψ are formulae of $\mathcal{L}_=$ then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$ and $(\phi \leftrightarrow \psi)$ are formulae of $\mathcal{L}_=$.

(iii) If v is a variable and ϕ is a formula then $\forall v\phi$ and $\exists v\phi$ are formulae of $\mathcal{L}_=$.

(iv) Nothing else is a formula of $\mathcal{L}_=$

Satisfaction in $\mathcal{L}_=$: As in the definition of satisfaction in $\mathcal{L}_2$ with the additional clause

(ix) $|s = t|^{\alpha}_{\mathcal{A}} = T$ iff $|s|^{\alpha}_{\mathcal{A}} = |t|^{\alpha}_{\mathcal{A}}$